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THE MATHEMATICAL GAZETTE

EDITED BY
THE LATE W. J. GREENSTREET, M.A., AND PROF. E. H. NEVILLE, M.A.
WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

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W. J. GREENSTREET.

I.

My recollections of W. J. Greenstreet date back half a century when we were both undergraduates at Cambridge, but we did not become personally acquainted till many years later. He had a slightly stooping figure and kind protective face which dissolved any feeling of awe produced by his great strength and height, penetrating eyes, and full-grown beard. He had two inseparable friends, E. F. J. Love and G. F. Stout. When in company together they drew the attention of every one who saw them. They looked more like three generations than contemporaries, Greenstreet being plainly the responsible head and Stout the cheerful but inscrutable infant, while Love appeared to be more normal and rather embarrassed by the strangeness of his companions. It was natural that such a remarkable-looking trio should receive a nickname; so they became known as the Three Graces. Too soon the inseparables were to become separated, each to make his mark in his special province; Greenstreet in Mathematics, Love in Science and Thermodynamics, and Stout in Classics and Philosophy.

It was a common interest in the *Mathematical Gazette* that brought Greenstreet and myself into touch with one another. At the beginning of 1896 I agreed to take charge of the *Gazette* until a permanent successor to its editor and founder, E. M. Langley, could be found. I do not remember how it came about; but very soon I was relying on Greenstreet for untiring help and advice. In July, 1897, he wrote that he would help in any way the Council thought advisable; and a year later he yielded to the pressure of myself and his friend Professor Lloyd Tanner to accept the editorship. From that time he carried on the difficult and heavy work till his death, more than thirty years later. As the circle of his contributors increased he gradually succeeded in making the *Gazette* the most readable and interesting of all mathematical journals. When he first undertook it, and for a further dozen years, he was Headmaster of the Marling School, Stroud; and I used to spend week-ends there from time to time. It was a joyous household. His wife was one of the brilliant family of Spenders, and contributed under the name of Aunt Medina the fashion articles to the *Daily News*, while Greenstreet supplied the Science Notes to the *Westminster Gazette* so long as it remained an evening paper. The rest of the household consisted of the school and family nurse, who afterwards became his second wife, and two small children. All these pursued their

individual aims and experiments independently of the rest, but formed a most harmonious whole. The *Mathematical Gazette* and methods of teaching were regarded as queer but harmless subjects of conversation in which they had no share. Greenstreet was specially keen on oral teaching and quickness of answering, a natural outcome of his own quickness and versatility. He was writing an Algebra in collaboration with Lloyd Tanner in which his special methods were elaborated; but, so far as I know, it never saw the light, and probably was never completed. With his wife he also made a not inappreciable addition to their joint income by solving prize competitions, acrostics and others; the same taste found an incessant and more professional but unremunerated satisfaction in the mathematical columns of the *Educational Times*.

A great tragedy befell him in the summer holidays of 1903, when his wife was drowned in his sight in a heroic and unavailing attempt to save her maid. She was a good swimmer but knew full well what a risk she ran from a weak heart. The remembrance of that day was written on his face for the rest of his life, and seemed to lurk in the sound of every word he spoke. It was followed by years of great despondency in which all interest in life had vanished. There were times when he was in great danger of falling into a state of melancholy; but he was saved by forcing his mind into the ordinary routine of his duties as headmaster and editor, and by a sense, whether conscious or not, that his mission in life was not yet completed. Hard work and a deep concern for others finally brought him round to a normal state once more.

Greenstreet did not fail to reach distinction; his name was well known to the whole mathematical world, and his monument was the *Mathematical Gazette*; but he did not reach a position to which his merit and ability entitled him. Luck was against him; his chance never came; and he was content. At the age of fifty he found that his ideals for his school were in opposition to those under whom he held his appointment, and in order not to sacrifice his freedom he resigned. He took a house on Burghfield Common in the midst of a fir-clad country, a few miles outside Reading; and there continued his literary work, lectured to pupil teachers, and applied himself with ever-increasing ardour to the editing of his favourite *Gazette*. His knowledge of what had been written in the range of modern elementary mathematics was probably greater than that of any other living person. He not only remembered what had been written on any topic but could lay his hands at once in his library on the volume where it was to be found. His memory and rapidity and ease of literary expression were such that he seldom had to make any alteration in the first draft of the articles and reviews he wrote. His conversation was full of humour, always with a warm and sympathetic ring in it; while everything he said was exhilarating and easy of apprehension. He was a great friend. He asked for and gained one's complete confidence. We discussed all our hopes and fears, successes and failures, and in any case of doubt or trouble his counsel was always wise and freely given. While one felt that he would himself be prepared to take risks regardless of consequences he would try to dissuade others from doing so.

Mention ought not to be omitted of the other inmates of the house on Burghfield Common. The chief of these were Sohrab and Rustum, twin pugs, who behaved with great decorum, and understood all that their master said to them. He was their benevolent ruler and protector. They formed his body-guard when he went a stroll; and came out to receive him when he returned from longer journeys. There was also an advance guard, a little black dog of incredible activity and springy lightness, the only one of its kind I have known.

His devoted wife survives him; also his son, Surgeon-Commander B. de M. Greenstreet R.N., and his daughter, who spent her energy and strength and impaired her health in the cause of her Country.

F. S. M.

II.

Reprinted, by kind permission, from *Nature*, Aug. 2, 1930.

By the death of William John Greenstreet on June 28 the mathematical world loses not an explorer or a geographer, but, if the metaphor may be pressed, a traveller familiar with a larger variety of landscape than almost any of his contemporaries. Born in 1861 and educated at St. John's College, Cambridge, he was an assistant master from 1882 until 1889, and headmaster of Marling School, Stroud, from 1891 until 1910, when he retired to Burghfield Common, near Reading, with the intention of devoting himself to literary work. For many years he had been a regular contributor to *Notes and Queries* and to the *Westminster Gazette*, and editor of the *Mathematical Gazette*, and he had every reason to anticipate a life congenial to his frugal tastes.

The War, however, put an end to Greenstreet's work for the *Westminster Gazette*, and at the same time raised the cost of living to an unforeseen level. The result was an extension of an activity which he had created, namely, the supervision of the education of pupil-teachers in village schools; the value of this work, which he had begun voluntarily in the schools nearest to his home, came to be recognised by the county educational authorities, and soon from 80 to 90 students looked to him for guidance, and he was known throughout the countryside, reaping in labours which earned him a livelihood the reward of spontaneous help given to his neighbours in happier years.

Meanwhile his editorship of the *Mathematical Gazette* continued, and it was this which made Greenstreet's name familiar to every mathematician in England. While the *Gazette*, as befits the organ of the Mathematical Association, has been concerned primarily with problems of school teaching, from elementary arithmetic to scholarship analysis, the characteristic features of the journal have revealed the editor. Greenstreet always desired to attain, and believed that all teachers benefit if they can attain, to such appreciation of current advances in mathematics as is possible without intensive study of special branches; he therefore encouraged ample notices of treatises, Continental and American as well as English, far beyond the range of school mathematics, until the review pages of his *Gazette* were admitted to be among the best in the world. Also, he had an immense knowledge of the personalities of literary, scientific, and social history, the product of omnivorous and rapid reading and a retentive memory; one result was that his own reviews of historical works, now tracing cross-currents of influence, now bringing a dead name to life by an anecdote or an epigram, enriched alike the books with which they dealt and the journal in which they appeared; another result was that every spare corner of the *Gazette* was filled by a 'gleaning', some quaint incidental reference to mathematics or to a mathematician found perhaps in classical literature, perhaps in a daily newspaper. In short, Greenstreet gave a character and a standing to a periodical which might have become nothing but a pedagogical mouthpiece; and this was the achievement that was acknowledged when the completion, in 1929, of thirty years of his editorship was the occasion of a testimonial to which some two hundred mathematicians subscribed.

Of Greenstreet's literary and musical interests it is impossible to speak here, but mention must be made of his enthusiasm for De Morgan. Once he was addressed as the De Morgan of his time, and this compliment pleased him as no other ever did. In wealth of biographical and bibliographical knowledge each was indeed unrivalled in his day, and this was perhaps all that the comparison was intended to convey, but one may recognise also in the two men the same sense of honour and the same sense of humour. Of the multitude of correspondents and contributors who were grateful for Greenstreet's help and counsel, few could claim to know him personally. His friends hold the memory of a man who never spoke a wounding or complaining word, of one

who was prodigal of his knowledge, forbearing in his judgments, and ready with his laughter.

E. H. N.

[The Editor's reviews were not confined to historical works. They were not signed, but they are easily identified; moreover, initials are attached in the Indexes to the volumes. From first to last, Greenstreet wrote just over five hundred reviews for the *Gazette*, as well as many reviews for other journals, and it can safely be said that not one of this multitude was perfunctory.

One of his finest reviews was never printed. Commissioned to deal with D. E. Smith's edition of the *Budget of Paradoxes* for the *Monist*, Greenstreet threw into a masterly essay the accumulation of a lifetime's absorption of De Morgan. The contrast between the reviewer and the editor of the book was too glaring, and the editor of the *Monist* declined to publish the criticism. Yet it need hardly be said that there was not an ill-natured word in the rejected notice: "The greater part of my remarks are not intended as a serious disparagement of what he has done . . . , but are rather a jeremiad at what he has left undone. . . . Discontent arises rather from the feeling that a great opportunity has been lost; it is in its essence of the nature of a subtle compliment."]

III.

It falls to me, as one who knew more than any other, to pay a tribute to one side of the work during the last twenty years of his life, of William John Greenstreet. It was after the great catastrophe of his life that he came to live at "The Woodlands" as a near neighbour. The first one heard of him was that he had undertaken to help the monitors and supplementary teachers of the parish schools to attain some qualifications which would benefit them in their career. He had himself been a victim of the new regime and, although he never spoke of it, was considered by others to have been very badly treated. As far as one can gather, the Governors of the Marling School would not agree to the demands of either the Board of Education or the L.E.A., and he had to suffer.

When he began life again in Berkshire, he was chiefly busied with the *Mathematical Gazette* and reviews for the *Westminster Gazette*, and he worked from early morning without any break until night.

Yet he found time to do innumerable acts of kindness and was at the service of anyone in trouble, young or old, to say nothing of dogs and other animals. He and his wife would give up their time to nurse and help the sick, and they nursed P. E. B. Jourdain in his last illness.

Since 1912 Greenstreet's work was, in the main, examination for County Scholarships, Bursarships and Pupil Teacherships, and the supervision and tuition of rural pupil teachers. He had from seventy to one hundred pupils in forty to fifty remote schools, and the devotion he displayed was amazing. He not only advised, but set weekly papers and corrected them, and saw each pupil at any convenient centre once a month. The results were remarkable, for the standard of Arithmetic and Essay-writing attained by these youngsters was easily in advance of that of a good secondary school. Their English Literature was also excellent, for they had learnt the great secret of working and thinking for themselves. Greenstreet had the gift of winning the confidence of the Head Masters and Head Mistresses, and they co-operated willingly and were glad to have opportunity of showing that they could rise above the routine of the antiquated Seven Standards.

The best proof of their success was not the mere examination results but the fact that the rural teachers, who went to College somewhat diffidently, came out at the top or near the top of their year. When they came back to the schools they were welcomed, and their early training showed itself at

once. They knew all about the atmosphere of the country school and could start work without stage fright or nervousness. Nothing gave Greenstreet more satisfaction than the letters from his old pupils now well settled and some in very good posts.

For the last three years, since the Board of Education decided to give up the old Preliminary Certificate examination, there has been much re-organisation necessary, but Greenstreet did not live to see the results of the new Oxford Senior Locals (Rural Teachers), when twenty succeeded in getting their qualifications under the new rules. This was what he had been working for.

The teachers so far have not had opportunity of testifying their appreciation of Greenstreet's work and their love for him, but every one that I have met since his death wished me to find some way of expressing it. It is hard to do, for an *epicedium* is beyond my powers and my sense of loss is great. "Quis desiderio?"

W. C. F. ANDERSON.

GREENSTREET AND THE *WESTMINSTER GAZETTE*.

FOR more than twenty years W. J. Greenstreet was scientific correspondent of the evening *Westminster Gazette*, and he was one of the first to bring scientific developments regularly under the purview of a newspaper. His work consisted in writing articles on any discovery or research of special importance, contributing a regular fortnightly article on current science, and reviewing scientific books and periodicals. He read widely, English, American and foreign books and scientific records of all sorts, to equip himself for this work, and, whenever possible, gave it a practical application. Though his approach to science was mathematical, he was keenly alive to its human and philosophical aspects, and wrote in such a way as to interest the ordinary reader, while catching the eye of the expert and the industrialist on the look out for new processes. Long before such advice became general, he was early in the field advising British manufacturers to employ more chemists and research workers and predicting that they would suffer, if they let their competitors in Germany or elsewhere run ahead of them in this respect. I had much evidence in these years of the interest which his articles excited, and though he covered a great deal of ground which was necessarily outside his own sphere, I cannot remember any occasion in which he was caught tripping by an expert. His scientific habit of mind saved him from the rash statements and presumptions which sometimes deface popular scientific journalism; and there were occasions when he firmly resisted the editor's call for pronouncements on new theories or discoveries on which he did not see his way. It has been a great advantage to science that it should kindle public interest and be competently handled in the modern newspapers, and Greenstreet deserves to be remembered as one of the pioneers of this kind of journalism.

J. A. SPENDER.

GREENSTREET AND THE LIBRARY.

NOBODY who has paid attention to the acknowledgments of gifts to the Library during the past seven years will have failed to observe that Mr. Greenstreet was the most lavish and the most persistent of benefactors. The records of the books of which he, as he put it, unloaded his own shelves as soon as the Library was removed to Reading, occupy several pages of vols. xi and xii of the *Gazette*; his name is in many of the volumes which were in the Library before that time, and few months have passed since without evidence of his goodwill.

The books which he gave away were not as a rule spare copies: he had very few duplicates. But I remember one occasion on which he had found that he had two copies of the same work, one perfect, the other complete but shabby.

The next time I called, he said, "The Library must have one of these," and then without a moment's hesitation, "And of course one can not give away one's second-best."

Only a few months before his death he gave the Association a few rare books, not mathematical, to be sold, as a contribution to the heavy expenses of binding the pamphlets and papers catalogued in the *List* which he had been helping to prepare.

Under his will the Association comes into possession of his runs of periodicals, of his early mathematical books, and of his historical, biographical, and bibliographical works of mathematical interest, and also of some other volumes and of a collection of offprints and papers. Members will not need the details, of which the preparation is necessarily slow, to know that from this bequest the Association derives a memorial which is valuable as well as appropriate.

E. H. N.

THE GREENSTREET TESTIMONIAL.

THE total of the fund which was opened on the occasion of the 200th number of the *Gazette* was £270. Of this amount, £200 was given to Mr. Greenstreet in June of last year, and the balance, with a list of subscribers, in May of this year, when it became all but certain that the last contribution had been made. From the first it was impressed on Mr. Greenstreet that the desire of his friends was to add to his personal comforts. He arranged at once to take a much needed and much enjoyed holiday in the West country, a holiday of which he had previously been afraid to incur the expense.

We are glad to be able to assure subscribers that throughout the last year of his life Mr. Greenstreet was continually utilising their gift in little ways. We are able to say also that when he read the list of subscribers, which included names of men and women in all ranks of mathematics, even the most distinguished, it gave him intense pleasure to feel that the work to which he had given so much of his later years was so generally appreciated.

That no acknowledgment of the testimonial was made in the *Gazette* was not due to any lack of gratitude or gratification. Mr. Greenstreet's first impulse was to write a paragraph for the October number last year, but he decided to wait until the fund was closed. Immediately on receipt of the closing cheque he sent to the printers a letter to appear in the July *Gazette*, but he grew dissatisfied with this general expression of thanks, and decided in place of it to send a modified letter to each of the 197 subscribers. A letter from the printers, dated 16th June, contains the paragraph:

"We thank you for your instructions that the letter referring to your testimonial is not to appear in the Correspondence and that a separate letter will be issued later. We have cancelled this matter accordingly."

This was within a few days of his last illness. We have not come across any draft of a modified letter.

E. H. N.
C. P.

GLEANINGS FAR AND NEAR.

777. Last year, we believe, the two mathematical almanacs, the Lady's Diary and the Gentleman's Diary, were united: the Gentleman being about 120 years old, the Lady about 140. We take an interest in these publications, in which so many English mathematicians have commenced their career. The joint publication, we think, ought to state the age of the parties in the title-page, as the individual ones used to do: is it that the Lady objects to appear as twenty years older than her husband? . . . The joint work is "designed principally for the amusement and instruction of students in mathematics; comprising many useful and entertaining particulars to all persons engaged in that delightful pursuit."—*The Athenaeum*, 1843, p. 1107.

GUNNERY: AND SOME OF ITS MATHEMATICAL PROBLEMS.*

By Prof. W. M. Roberts, M.A.

The title of my lecture is rather misleading. When I was asked to deliver it I thought I could profitably indicate how very definitely mathematics is used in a particular activity, such as gunnery, the application being neither complicated nor varied. If one was asked to write a paper on the utility of mathematics in engineering or in wireless electricity, the problem would be much more diffuse. The man who makes or fires a gun is only, after all, making one particular machine or doing one particular job. Hence it is possible briefly to state what mathematical difficulties he encounters.

But as I thought the matter over and looked up a few things I came across an early treatise on gunnery by a certain Nicholas Tartaglia, who lived before the days of Newton, and reading that treatise, it struck me that some of the matters coming to light in that book were of definite use to people like ourselves who are mostly teachers. It brought out clearly the difficulties of the practical man before Newton's great generalisation had cleared up the science of dynamics, and in those difficulties the interest of the teacher comes in. The average pupil is groping in the dark much in the same way as the pre-Newton mechanician. Tartaglia's troubles and failures were not altogether unlike the troubles met with by a girl or boy struggling with mechanics in its early stages. Hence, although I shall keep to some extent to the subject as defined by the forecast of my lecture, I will make no apology for asking you for a few minutes to try and put your minds back before Newton's laws, to forget them, and to forget much more the idea of conservation of energy—in a word, to try and use intuition when the facts are not selected for you by your teacher.

Newton's great triumph always seems to me to have been that he isolated the force and the accompanying acceleration from an already existing motion—for example, when he broke up the parabola in unresisted motion into a pre-existing velocity and a constant acceleration, and yet got his curve all right; or again, when in circular motion he did the same thing with a pre-existing velocity and another acceleration still constant, but this time varying in its direction.

Nicholas Tartaglia was the inventor of the first solution of a cubic equation, and I want to suggest to you that he was a very acute man. The treatise referred to was published in 1537, and translated by one Lucar into English in 1588. It is called *A Book of Colloquies on Artillery*, and the argument is in the form of a conversation with a certain duke and a prior. The book is an excellent example of reasoning in the days when scientific habits of thought were not prevalent. Tartaglia's first conclusion was a very correct one—he had evidently been doing a little bit of experimenting—namely, "A piece of artillery laid at one point† will shoot more further than when it lieth level," at two points more than at one point, at three points more than at two, and so forth, but at seven points less than at six points, and at eight points less than at seven. He adds that at twelve points it might be thought by general reason that the pellet should return back again into the mouth of the piece, but very many accidents "which do commonly happen at that instant when the piece is discharged" cause the pellet not to return precisely into the mouth of the piece, but, he says, very nearly to do so. But the gunner may make a table of ranges.

Thus he anticipates the laws of casual error, but he does not specify what

* A lecture delivered on January 7th, 1930, at the Annual Meeting.

† A point is 7½°.

the many accidents are. The following conversation with the duke then ensues :

" The Duke : What is the good of this ?

" Tartaglia : The knowledge of this will be a secret.

" The Duke : But if the man with the table shows one how to shoot, the latter will learn the secret.

" Tartaglia : No, the other person is like the servant of the apothecaries who continually propounds medicines as appointed by the physicians but does not thereby become a physician."

All the same the duke was right.

The next problem I came across was the following : Will a gun laid level strike a target harder than a gun at a level below ? The author says, " Without doubt that piece laid on the plane below will be the more effective." The duke states that he would expect the contrary, to which the author replies, " No, a piece which lieth level will never shoot so far in a right line as when it was somewhat elevated," and he adds the following note, " A piece will shoot farther in a right line when the mouth doth lie a little downward." Then the dialogue proceeds :

" The Duke : This is a strange and incredible tale, that the same quantitie of powder will expell the more violently one and the same weight of pellet."

" Tartaglia : The science of weights, which science is of no small speculation, says the contrary."

Tartaglia says in effect, making thereby a very serious statical error, that if a rod with equal heavy weights at the ends is pivoted about its centre, the weight of each body is decreased as the rod is turned about that centre ; really he is mixing up the weights and the tension in the rod and concluding that in some curious way the weight is itself affected by the obliquity of the rod, for he definitely deduces that " the body is lighter at one point than at another," so that a " pellet flew more heavily from a piece lying level ". This explains for him why a pellet will shoot 200 paces when lying level, but at 6 points (45°) it will shoot 800 paces. He has the idea of kinetic energy and work, but alternately rejects and accepts it.

His argument now proceeds. A gun is laid level at A and shot horizontally, striking a wall 100 paces away at B. Since the range is 200 paces, the shot has still 100 paces to go when it hits B. But if laid at 45° to hit the wall at C, since AC is about 140 paces, and the range at 45° elevation is 800 paces, the shot has still 660 paces to go after it struck C, hence it must have struck at C much harder than at B !

Tartaglia has occasionally curious glimpses of light. He says, " The great swiftness is the proper cause to make the pellet fly in a right line, if possible, and likewise the lack of swiftness is the proper cause to make it go and decline crookedly in its way toward the ground, and by how much the swiftness decreaseth, by so much it maketh his declination or crookedness the greater . . . because every heavy body driven violently through the air, by how much the swifter it fleeth, so much, in moving, it is made the more lighter, and therefore it goeth the more rightly in the air." The lighter a body is the more easily the air will bear it. Again, he says that the more weight there is in the pellet the more there is a provocation to shoot the said pellet towards the centre of the earth. By all this verbosity he manages to prove that if you shoot the gun at a particular point the path cannot be straight—the pellet is swifter here than it is there. He is making a great deal of argument for what is really a truism, yet later he makes this correct remark that " the weight draweth the pellet out of his way perpendicularly to the ground." He noted, quite correctly, that very often the second round goes further than the first because, he says, " the air is moved," but if you keep on shooting the range falls off. He settles that by saying that the air is moved and so the second round gets through more

easily! He also says that the later shots fall shorter when the piece is hot. This is explained by "attraction in the manner of a cupping glass"! The gun sucks in more and more of the exhalation of the effect of the saltpetre. His next error occurs when he says that if you aim at a moving ship you do not damage it so much as you would a stationary wall, the reason alleged being that the ship is moving. He goes on to make a curious statement that not so much damage is done to an object close to a gun as to an object further away. That is explained by the fact that every body moves something else, therefore air is carried in front of the pellet, this air, rebounding back, strikes the pellet, so that one does not get hit quite so hard if one is nearer to the gun as one would at a further distance.

He then relates what he calls the bombardier's story. "After many shots a certain gun rose up so that the mouth went into the ground. While I was busy providing labourers to use levers to get the piece into his place, a little dog going by did smell into the mouth of the same piece, and by so doing was suddenly joined to the mouth of the piece, and immediately afterwards drawn into the concavity of the said gun: which things when those standing by had seen, some ran to help the said dog, and although they perceived him to be drawn even almost to the farthest end of the concavity, they pulled him out almost dead. What became of him afterwards, I know not, but I think he died." Nicholas explains this by "the cupping effect," and the duke remarks, "Your reasons please me well."

Just one other little point. This old author is rather worried by the way in which noise arises. "The noise," he says, "is due to the exhalation of the wind caused by the saltpetre leaving the piece in the air. Sound is no other thing than the blow which two bodies that have no souls or lives do make together."

Lastly he goes on to show that if the metal of a gun is thicker one side than the other, the gun will shoot to the thicker side, a clever observation.

All this divergence into an ancient treatise may be irrelevant, but I thought it would be amusing if only to show you a good man struggling with difficulties.

There are one or two problems of gunnery, including some very elementary applications of trigonometry, which I should like to bring to your notice. You can get quite a nice dynamical problem out of a gun. Take the weight of the actual piece itself, the weight of the pellet, the muzzle velocity of the gun and the length of the bore, and those data will involve you in a pleasant little calculation as to the force on the shell, how far the gun recoils while the shell is in the bore, the thrust on the mounting, not accurate since one must assume a constant force unless one uses calculus methods, but near the truth. If you assume that the force is constant you can even make a fair approximation in terms of tons per square inch of the pressure on the base of the shell.

Let me now explain the gunner's job. Nowadays when a gunner goes into position the first thing done is to send out a party who lay out the position of a number of important targets and a number of intermediate targets by accurate survey. In laying the gun and calculating the position of the fall of rounds relative to the target, there are several calculations, all quite simple, to be made involving small angles. Thus if B is the gun, O an observer and T the target (which the observer can see and the gun cannot), when OB is small compared with BT, but known in length, a neat and simple calculation will deduce the angle at T, and thence that at B in the triangle TOB. Or again if two guns, a given small distance apart, are laid parallel, and number 2 gun wishes to concentrate on number 1 gun's target, a similar calculation occurs.

Again, if the wheels of the gun are not level, the wheel on one side being a little higher than the other, so that the gun is no longer swinging in

the plane you want, a little calculation is required to find the angle through which the line of fire is shifted.

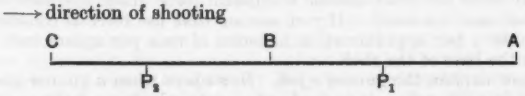
Having got the line of the gun, the next thing the gunner has to do is to get his range right, and that is not as easy as might be supposed. If a gun is placed in a certain position and a shell is fired, a subsequent shell will not necessarily strike at the same point as the first. A couple of years ago Mr. Hope-Jones read a paper to this Association on this subject. I will only refer you to that paper and will add that, roughly speaking, you can say that if you fired a certain number of rounds out of a gun, all at the same elevation, and the rounds neatly piled themselves up on the ground where they hit, you would get a heap shaped in a definite manner following the laws of probability.

How, then, do you hit a given target? That is rather an interesting application of theory to an awkward practical job. First of all we have to look at the shape of this curve. The way the gunner deals with the problem is as follows: he sees what I have called his little heap of shells. He takes two equal distances from either side of the mean point of impact so as to get a zone which contains half the shells, and this he calls the 50 per cent. zone. The 100 per cent. zone would be theoretically infinite in length, but in practice if you take four times the distance on either side you will include an area covering practically every shell. Out of every hundred shells 99·4 will fall inside that zone, which is four times the 50 per cent. zone. A zone one and a half times the length of the 50 per cent. zone contains 83 per cent. of the rounds. The gunner finds his range with a very ingenious application of the idea of these 50 per cent. zones. If anybody wants to work it out by mathematics, he can amuse himself with the curve

$$y = Ae^{-\frac{x^2}{h^2}}.$$

If the mean distance of the shells from the mean point of impact is measured and multiplied by 1·69, you get the length of the 50 per cent. zone. That is the way the gunner goes to work. The number 1·69 can be calculated by theory from the shape of the curve above.

It is well to remember that when shooting at a target you can rarely tell how far the missile goes beyond it. You only know that a round has gone plus or minus the target. But supposing you have your gun and shoot it off at a certain angle of elevation which would give you the mean point of impact at the point I call P_1, and you measure twice the 50 per cent. zone each side of P_1, you will know that practically every shell fired at that elevation will fall inside AB.



In what follows, think of the target relative to the shot, not the shot relative to the target. If the first round whose mean point of impact is P_1 is plus, the target must be short of A (ignoring the extraordinary rounds not inside AB). The elevation of the gun is now dropped by the equivalent of 4 times the 50 per cent. zone; this is called the "long bracket." If the M.P.I. of the next round is at P_2, we can measure the approximate 100 per cent. zone for P_2. This is BC, and if the second shell is minus, we know that the target is plus of C and minus of A.

Now raise the elevation so that the M.P.I. is at the mid-point of P_1P_2, *i.e.* at B. P_1P_2 is the 100 per cent. zone of B. If this round is, say, plus, the target lies between C and P_1, and we have now what is called the short bracket. The next step is to verify the short bracket, *i.e.* fire again at the P_2 elevation

and then at the B elevation. If these rounds are minus and plus respectively, the short bracket is "verified," and we know with fair accuracy that the target is rather plus of C, and rather minus of B, and so is very likely between P_2 and B. The calculation of the chance of this is a nice problem in probability.

If we now fire at an M.P.I. midway between P_2 and B, we are very likely to have our target in between P_2 and B, and so a series of rounds is fired at this new elevation. If over this series there is a fairly even distribution of plus and minus, the target must be close to the mid-point of P_2B.

There are many interesting problems in probability connected with this theory of zones. Theory tells us that the ratios between the various percentage zones are fixed. Thus the 83 per cent. zone is one and a half times the 50 per cent. zone, and so on.

Suppose the 50 per cent. zone is 40 yards, and the target is a strip of ground which would receive 8 per cent. of the rounds if the M.P.I. were exactly in its centre, O. Now assume that the M.P.I. is 20 yards (half the 50 per cent. zone) short of O. It can be shown that about $6\frac{1}{2}$ per cent. of the rounds would fall in the given strip, not a very great difference.

It is not worth while altering the gun by very much more than half the 50 per cent. zone. One of the gunner's rules, based on calculation, is that it is not worth while altering the gun by small amounts. This merely supplements what Mr. Hope-Jones had to say to you about probability a couple of years ago.

Now something about the range of the gun. The range table gives you the information. When you speak of the range for an angle of elevation of 5° you mean that that is the range to the mean point of impact. In unresisted motion there is no difficulty about finding the range. If you can find the muzzle velocity of a gun and if you shoot on a perfectly flat plane, a rifle bullet in unresisted motion could go about 50 miles.

Prof. **Lodge**: You could not get a perfectly flat plane of 50 miles. The curvature of the earth would come in.

The **Lecturer**: That is true. I was discussing the theoretical consideration. If we take a 6-in. howitzer which has a low velocity, the howitzer would reach 15,000 yards without air and would only reach 10,000 yards with air.

There is a certain formula in this connection known as Sladen's formula, which is as follows:

$$h = \tfrac{1}{2}gtt',$$

t being the time from the start to a point, t' the time from that point to the target and h the height of the point for a parabolic trajectory; this formula is easy to prove.

Shooting in a very flat trajectory, as with a field gun, which does not as a rule shoot very much above 8° or 9°, and is generally below 5°, the forces acting on the projectile are the weight and the resistance of the air. The vertical force is $W + R\sin\theta$, where θ is the angle the direction of motion makes with the ground, and R is air resistance. For a great part of the trajectory θ is so small that $R\sin\theta$ is negligible compared with W. Hence the vertical motion of the projectile in the real trajectory is precisely the same as that in the trajectory *in vacuo*. For that reason Sladen's formula applies to this real trajectory as well as to the imaginary trajectory. Supposing we want to shoot a gun at a range of 4000 yards, and we want to find the height of the trajectory at 1000 yards, we first of all look at what the range table gives us. We look up the time of flight at 4000 yards. The range table will give us the time of flight for 1000 yards, but I think it is fairly clear that this is t. Therefore if you look up t for 1000 yards and for 4000 yards you can get t and t', and the height for low angles. That is an exceedingly good method for finding the height of the trajectory at any point in its path.

When the angles are small the trajectory will behave as if it were exactly rigid when shooting slightly uphill, so that the formula which is true *in vacuo* is also so nearly true that it does not signify in a real trajectory where the angles are small. If you want to shoot up an inclined plane, *in vacuo* when the angles are not small, a formula may be deduced connecting the various angles involved. You would not expect that to be even approximately true when the angles are large, but, in point of fact, so long as you do not get up above 40° you will not be far out on a real trajectory. Such a formula was actually used at one time in the war when they started having bursts of shell high up in the air observed by two people simultaneously so as to locate exactly where the bursting took place. I have thus given you two very good examples of the value of parabolic motion.

I want to say a further word or two about air resistance and its difficulties. The law of resistance of the air is not known in any compendious manner. You can observe the resistance to a given body at a given speed. Supposing there were such a law as $p=Kv^n$, the equation of motion can easily be written down.

You can easily see that the solution is going to be complicated. All sorts of ingenious plans have been suggested. The most amazingly ingenious seems to be that of using θ, the direction, as an independent variable. Even if there were a simple law, you cannot solve your differential equation. But there is no law like that, you can only approximate on the lines that for certain values of the velocity n is approximately so and so, and for another series of values it is so and so. In other words, you plot $\log v$ against $\log p$, getting a varying curve, and you treat certain portions of it as being straight. You get appalling equations, and there is nothing to do but numerical integration. The general idea is that there is a table which gives the resistance of the air to every velocity for a shell whose diameter is 1 in. It is always assumed, but never actually proved, that if the diameter is increased to D the resistance of the shell becomes pD^2. That is very nearly true, though not absolutely true. The density of the air comes into the matter, and various other things have to be allowed for, so that the upshot of the whole thing is that you have a certain constant C, called the Ballistic Coefficient, which makes allowances for the barometer and so forth.

If you have got the values of p you can extract from your table the value of the resistance to the standard projectile at various velocities. Using the equation $\Delta t=\Delta v/Cpg$, easily obtained, with Δv equal to 10, and referring to the table for p, the time taken for the velocity to drop by 10 f.s. in the neighbourhood of a selected velocity can be calculated, and so, laboriously, the time taken for any given drop in velocity.

If you had to estimate the range of a shell, say, at 5° elevation with a muzzle velocity of 2822 ft. and a weight of 355 lbs., unless you had the range table to tell you, it would take you many hours of arithmetic to work it out, even with a flat trajectory. When the angle of elevation is not small, difficulties increase. What you do is to take a small piece of the arm of the trajectory that will be covered by the shot in a fraction of a second. Various things can be taken as sensibly constant during that part of the trajectory, and from this you can tell fairly well where the thing is at the end of one-fifth of a second. You then start again and with a new velocity and new angle of elevation repeat the calculation, and so on step by step for possibly a time of over 40 seconds, meaning 200 calculations, and by that means you eventually find where the shell hits the ground. But this is not work for anyone but the specialist.

Prof. **Lodge** proposed a vote of thanks for a most interesting address, and it was carried with acclamation.

SOME EXAMPLES OF THE TWO-DIMENSIONAL MOTION OF A LIQUID.

Communicated by S. F. GRACE, D.Sc.

§ 1. The present communication forms, in essentials, part of a paper read before the Liverpool Mathematical Society in 1900 by Rev. S. Sircom, S.J., M.A. Formulae for the stream lines in the two-dimensional irrotational motion of a liquid were given in certain particular cases and illustrated graphically by means of a series of remarkable diagrams. These diagrams are unique; it was felt that their reproduction in permanent form was desirable and would at the same time prove of interest to readers of the *Mathematical Gazette*.

It is to be regretted that, on account of age, Fr. Sircom has not been able to co-operate actively in the production of this account; the writer is, however, much indebted to him for the interest he has taken and for the loan of MSS. and diagrams.

§ 2. It is proposed to examine some of the solutions which have been obtained of the following problem in the two-dimensional motion of a liquid: "To find the stream function when a cylinder, whose cross-section is a given curve, is moving in an infinite liquid and also when a liquid is streaming steadily past a fixed cylinder." On drawing the stream lines in some of the cases it will be found that a given curve may be a boundary in a case of liquid motion otherwise than as a section of a solid cylinder, so that the second part of the problem actually solved may be written: "To find the stream function in an infinite liquid in steady motion when some part of the boundary is a given curve."

Let u, v be the components of velocity for two-dimensional motion of a liquid parallel to the plane xy; then the differential equation of the stream lines is

$$\frac{dx}{u}=\frac{dy}{v},$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

which is the analytical condition that $u\,.\,dy-v\,.\,dx$ is an exact differential. Hence there exists a function ψ, the stream function, such that

$$u=\frac{\partial \psi}{\partial y},\quad v=-\frac{\partial \psi}{\partial x},^{*}$$

and for irrotational motion

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=0.$$

§ 3. The only rational form of ψ that has been obtained for two-dimensional motion is

$$\psi=\Sigma\mu\frac{(y-b)\cos\alpha-(x-a)\sin\alpha}{(x-a)^2+(y-b)^2},$$

which corresponds to a system of doublets, of which the typical doublet of strength μ is situated at (a, b) and inclined at α to the x axis†.

Taking two doublets, strengths μ, ν at the points $(\pm a, 0)$ and axes along OX, the stream lines are

$$c=y\left\{\frac{\mu}{(x-a)^2+y^2}+\frac{\nu}{(x+a)^2+y^2}\right\},$$

* The stream function is now usually defined by the relations $u=-\partial\psi/\partial y$, $v=\partial\psi/\partial x$.

† Ramsey, *Treatise on Hydromechanics*, II. p. 48.

where c is a constant which can take all real values; such curves will be referred to as the c-curves. If, in addition, liquid is moving with velocity V in the direction XO, the stream lines are

$$\psi_x = y\left\{\frac{\mu}{(x-a)^2+y^2}+\frac{\nu}{(x+a)^2+y^2}-V\right\}=-n,$$

where n has all real values; these will be referred to as the n-curves. The quartic curve

$$V\{(x-a)^2+y^2\}\{(x+a)^2+y^2\}=\mu\{(x+a)^2+y^2\}+\nu\{(x-a)^2+y^2\} \qquad (3.\ 1)$$

is a bounding curve of the series $\psi_x=-n$; over this curve $\psi_x=0$ and $c=Vy$. Hence the c-curves correspond to the stream function when the curve (3. 1) is moving with uniform velocity V through the liquid. Both c- and n-curves pass through the critical points of infinite velocity $(\pm a, 0)$, but if the relative values of V, μ, ν are so chosen that these points are enclosed within the bounding curve, the liquid velocity at all points outside the curve is finite and the n-curves represent the stream lines for liquid flowing past the cylindrical obstacle whose cross-section is the curve thus determined.

Particular cases are:

(a) $\nu=0$. This gives the well-known forms for the circle.

(b) $\mu=\nu=\frac{1}{2}Va^2$. Writing Vc for c, the c-curves are

$$c=y\frac{a^2(x^2+y^2+a^2)}{\{(x-a)^2+y^2\}\{(x+a)^2+y^2\}},$$

and, writing Vn for n, the n-curves are

$$n=-y\left[\frac{a^2(x^2+y^2+a^2)}{\{(x-a)^2+y^2\}\{(x+a)^2+y^2\}}-1\right],$$

or $$n\{x^2+y^2)^2-2a^2(x^2-y^2)+a^4\}=y\{(x^2+y^2)^2-a^2(3x^2-y^2)\},$$

and the bounding curve is the lemniscate form

$$(x^2+y^2)^2=a^2(3x^2-y^2).$$

The c- and n-curves are illustrated in Fig. 1.

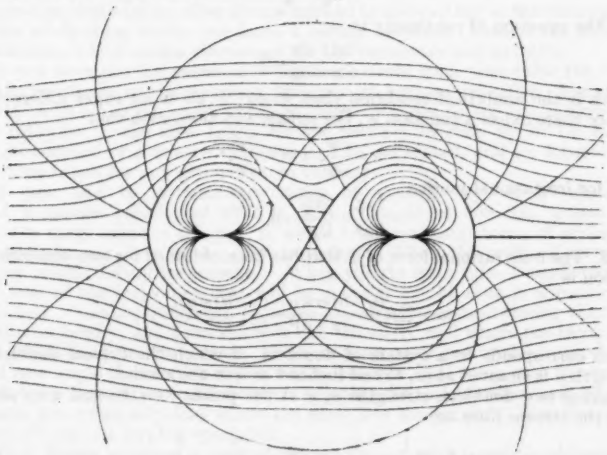


FIG. 1.

§ 4. Considering next the lemniscate of Bernoulli,

$$(x^2+y^2)^2=2a^2(x^2-y^2),$$

the stream function, when the lemniscate moves parallel to OX, is

$$\psi_x = -\tfrac{1}{2}Uai\left[\frac{x-iy}{\sqrt{\{(x-iy)^2-a^2\}}}-\frac{x+iy}{\sqrt{\{(x+iy)^2-a^2\}}}\right],$$

and, for motion parallel to OY,

$$\psi_y = -\tfrac{1}{2}Va\left[\frac{x-iy}{\sqrt{\{(x-iy)^2-a^2\}}}+\frac{x+iy}{\sqrt{\{(x+iy)^2-a^2\}}}\right].*$$

When cleared of imaginaries the c-curves for ψ_x may be written

$$c^4\{(x^2+y^2)^2-2a^2(x^2-y^2)+a^4\}^2$$
$$+c^2a^2\{(x^2+y^2)^2-2a^2(x^2-y^2)+a^4\}\{(x^2+y^2)^2-a^2(x^2-y^2)\}-a^2x^2y^2=0,$$

no extraneous solution being introduced by rationalization. Taking $a=8$ the curves in Fig. 2 have been constructed from this equation for $c=\pm(1, 2, 3, \ldots)$. Near the apses of the lemniscate the curves do not differ greatly from the corresponding curves in Fig. 1, but near the node they are entirely different and $c=0$ gives OY as well as OX as a limiting curve. The liquid motion may be interpreted as follows: In the ordinary conception of a source the issuing liquid spreads equally in all directions. Consider, however, a source such that the velocity at the orifice is in a definite direction; this may be supposed to be secured by the liquid issuing from a small tube in the plane of the motion. Let there be two such tubes, one corresponding to a source just above OX and in that direction, the other to a sink just below OX in the same direction, the orifices being at S (Fig. 2) with a barrier SX to prevent liquid flowing directly from one to the other, then if there be a similar pair at H on the other side of O the stream lines will be the curves in the diagram. The stream function for two such doublets, or for one when a line perpendicular to the directed velocity is a boundary, have therefore been found indirectly†.

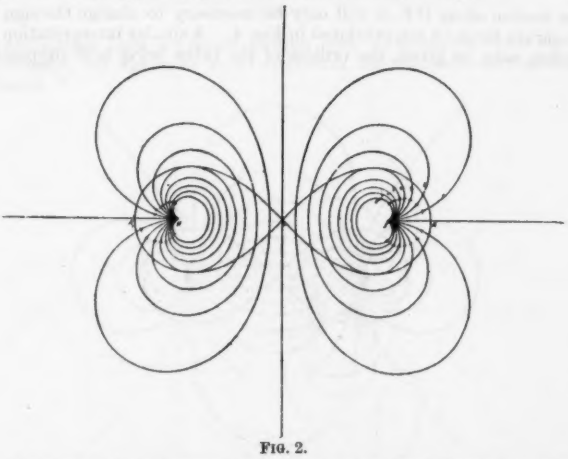


FIG. 2.

The n-curves for liquid streaming past are obtained by writing $y-n$ for c. As in the case of the c-curves they resemble somewhat the corresponding curves in Fig. 1 near the apses, but instead of the depressions near the node there are double points along OY. The curves are symmetrical with respect

* Basset, *Treatise on Hydrodynamics*, I, p. 109.

† Prof. A. E. H. Love has remarked that the c-curves in Fig. 2 are the lines of flow of the liquid when the two halves of the lemniscate are on the point of separating along OX and XO.

to both axes, but to avoid confusion only one branch of each is drawn and exhibited in Fig. 3. Beyond curve 8, which has double points at $x = \pm 4\sqrt{3}$, $y = 4$, the points of contact of a double tangent to the lemniscate, the curves break up into isolated ovals with infinite branches outside. In the region of the infinite branches the velocity will be everywhere finite and the curves there are stream lines for liquid flowing past the curve 8.

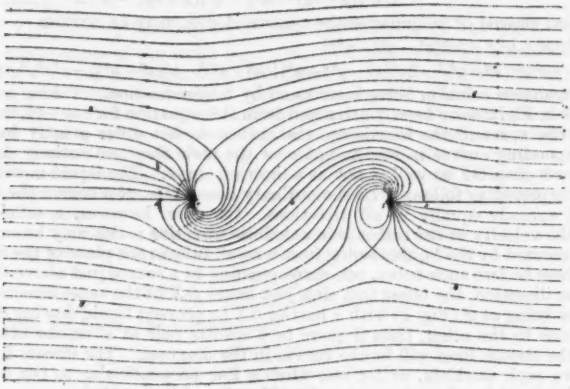


FIG. 3.

For motion along OY, it will only be necessary to change the sign of c^2 *; the c-curves for $a = 8$ are exhibited in Fig. 4. A similar interpretation to that preceding may be given, the orifices of the tubes being now supposed to be

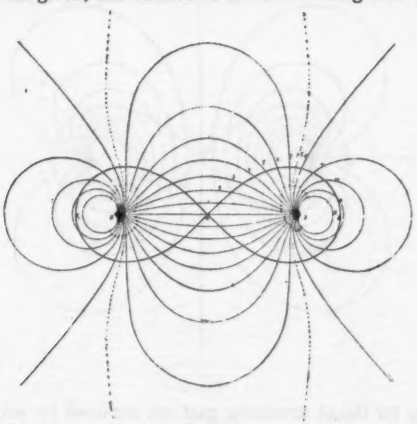


FIG. 4.

directed towards O. Outside the curve for which $c = 8$, which has the asymptotes $x^2 - y^2 = 0$, the liquid may be considered to flow from the source at S to the sink at S, but inside from S to H, SH being a barrier.

* For motion in a direction inclined at an angle θ to OX it would be necessary to replace c by $ce^{i\theta}$.

The n-curves are obtained by writing $n-x$ for c and are shown in Fig. 5. For $n=16\sqrt{2}$ the branch of the curve nearest O has a double point at the apse of the lemniscate; outside this the velocity is everywhere finite and the

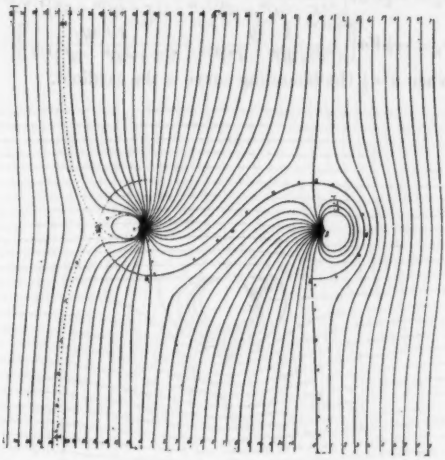


FIG. 5.

curves are stream lines for liquid flowing past. The curves cover the plane symmetrically, but again, to avoid confusion, the complete curves have not been drawn. *

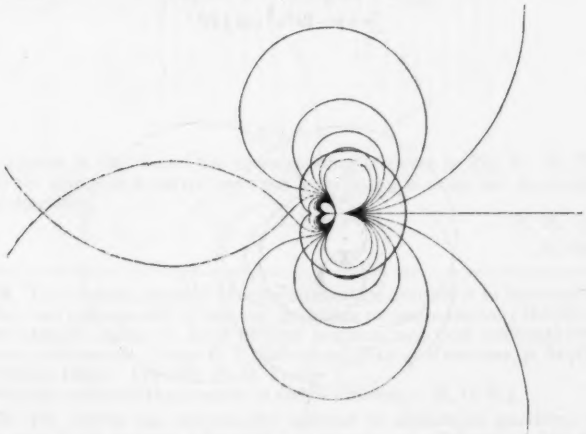


FIG. 6.

* In Figs. 2, 3, 4 and 5, S is the common cusp of the c- or n-curves on the positive x-axis, H that on the negative x-axis.

§ 5. The expressions for ψ for the lemniscate of Bernoulli are particular cases of the more general forms

$$\psi_x = -\tfrac{1}{2}Uai\left[\frac{x-iy}{\{(x-iy)^n-a^n\}^{1/n}} - \frac{x+iy}{\{(x+iy)^n-a^n\}^{1/n}}\right],$$

$$\psi_y = -\tfrac{1}{2}Va\left[\frac{x-iy}{\{(x-iy)^n-a^n\}^{1/n}} + \frac{x+iy}{\{(x+iy)^n-a^n\}^{1/n}}\right],$$

when $n=2$; when $n=\frac{1}{2}$ the bounding curve is the cardioid.

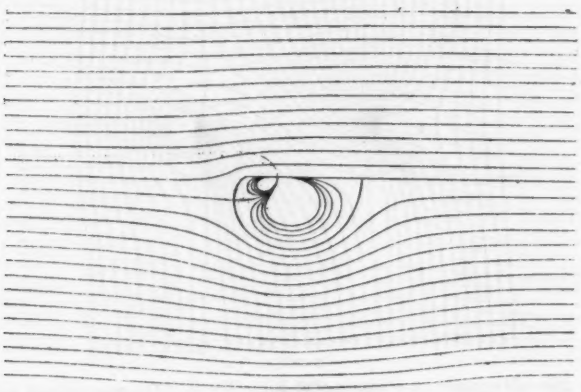


FIG. 7.

For the cardioid the c-curves for motion along OX may be written (in polars)

$$c = \frac{2ra^{\frac{3}{2}}\sin\frac{1}{2}\theta\,.\,(r^{\frac{1}{2}} - a^{\frac{1}{2}}\cos\frac{1}{2}\theta)}{(r+a-2r^{\frac{1}{2}}a^{\frac{1}{2}}\cos\frac{1}{2}\theta)^2};$$

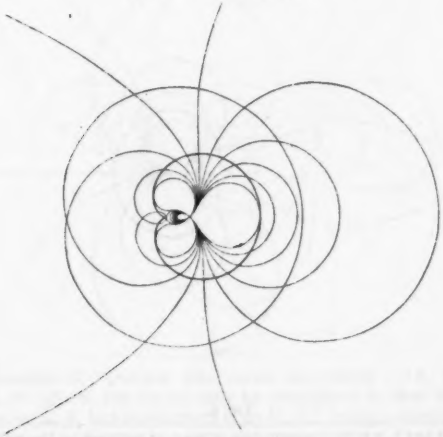


FIG. 8.

they are exhibited in Fig. 6 for $a=2$. The corresponding n-curves are obtained, as before, by writing $y-n$ for c and are shown in Fig. 7. For the motion at right-angles to the axis of the cardioid the c-curves have the equation

$$c=-\frac{ra(r+a\cos\theta-2r^{\frac{1}{2}}a^{\frac{1}{2}}\cos\frac{1}{2}\theta)}{(r+a-2r^{\frac{1}{2}}a^{\frac{1}{2}}\cos\frac{1}{2}\theta)^2};$$

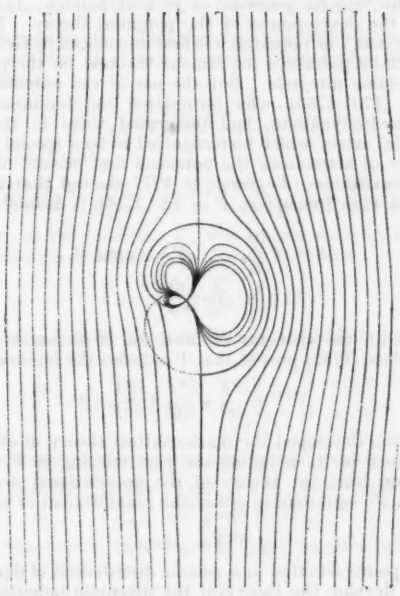


FIG. 9.

they appear in Fig. 8 and the corresponding n-curves in Fig. 9. In Figs. 7 and 9 the complete n-curves have not been drawn in order not to complicate these diagrams.

S. F. GRACE.
S. SIRCOM.

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778. To be wholly devoted to some intellectual exercise is to have succeeded in life; and perhaps only in law and the higher mathematics may this devotion be maintained, suffice to itself without reaction, and find continual rewards without excitement.—From R. L. Stevenson, *Weir of Hermiston*, p. 24 (Chatto & Windus, 1925). [Per Mr. H. G. Forder.]

[But the nature of the rewards is rather different.—H. G. F.]

779. The author has occasionally resorted to algebraical notation, which some may think scarcely allowable in a treatise on arithmetic; but we see no harm in it when it is done sparingly, and is really the simplest available method.—*The Course of Arithmetic as taught in the Pestalozzian School, Worksop*, by J. L. Ellenberger. Review in *Athenaeum*, 1854, p. 148.

THE USE OF SPHERICAL HARMONIC FUNCTIONS IN MATHEMATICAL PHYSICS.*

By PROFESSOR S. CHAPMAN, M.A., D.Sc., F.R.S.

1. The applications of spherical harmonic functions in mathematical physics are closely connected with the various potentials—gravitational, electrostatic, magnetic, or the velocity potential in hydrodynamics. It is of interest to recall, briefly, the development of the potential theory. The actual name "potential function" or potential was first introduced by Green in the memoir of 1828 in which he published his famous theorem on the transformation of surface and volume integrals. But the *use* of the potential function goes back to Euler (1707-1783), who formulated the fundamental differential equations of hydrodynamics, and integrated them in the case where a velocity potential exists, which he recognized to be a special case.

In the theory of attractions the potential first occurs, naturally, in connection with gravitation. Lagrange in 1777 showed that the gravitational force at a point P in free space, $\bar{F}$ or (X, Y, Z), is deducible from a scalar function V, *i.e.*

$$X = -\frac{\partial V}{\partial x}, \ldots, \quad F = -\text{grad } V,$$

where

$$V = \int \frac{dm}{r}$$

integrated over all the elements of mass dm, at distances r from P. Soon afterwards Laplace (1782) proved that V satisfies the famous equation:

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

In 1812 Poisson developed the mathematical theory of electrostatics, and showed that much of it, including the part relating to V, might be taken over ready-made from the theory of the gravitational field. In 1813 he extended Laplace's equation, in either the gravitational or electrostatic case, to the form

$$\nabla^2 V = -4\pi\rho$$

appropriate to points situated in a volume distribution of density ρ. In 1821 he went on to develop the theory of the magnetic potential, in a masterly memoir which brought out some important features of the magnetic case which have no simple gravitational analogue.

2. While the general theory of the potential was thus being developed, new mathematical functions were being introduced and studied, largely arising from special problems in the theory of attractions. One important problem, connected with the theory of the Figure of the Earth, was that of a gravitational

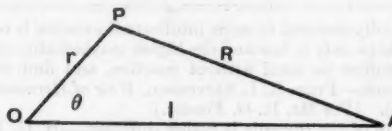

FIG. 1.

fluid globe, rotating or not, subject to the attraction of a distant body or particle A. In this problem it was desirable to express the potential at a point

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P, due to A, as a function of the coordinates r, θ of P, relative to the centre O of the fluid body, and the line OA.

Taking OA as unit distance, and the particle at A as of unit mass, for simplicity, it is necessary to expand the potential $\frac{1}{R}$ as a function of r, θ; now

$$R^2 = 1 - 2\mu r + r^2; \quad \mu = \cos\theta.$$

Clearly $\frac{1}{R}$ can be expanded as a series in r (convergent for $r < 1$), the coefficients being functions of μ. This was done in 1784 by Legendre, after whom the coefficients, denoted by $P_n(\mu)$, are called Legendre functions or Legendre polynomials. Thus

$$\frac{1}{R} = \frac{1}{\sqrt{1 - 2\mu r + r^2}} = \sum_{n=0}^{\infty} P_n(\mu) r^n,$$

and Legendre showed that

$$P_n(\mu) = \frac{(2n)!}{2^n (n!)^2}\left\{\mu^n - \frac{n(n-1)}{2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)}\mu^{n-4} - \ldots\right\}$$

3. The theory of the functions P_n was quickly developed. Other expressions for P_n were soon found, such as that of Rodrigues:

$$P_n = \frac{1}{2^n \cdot n!}\frac{d^n}{d\mu^n}(\mu^2 - 1)^n.$$

Legendre (1784 and 1789) discovered the remarkable integral properties of the P_n functions:

$$\int_{-1}^{1} P_m(\mu) P_n(\mu)\, d\mu = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2n+1} & \text{if } m = n. \end{cases}$$

Numerous recurrence formulae were also proved, such as

$$(n+1)P_{n+1} - (2n+1)\mu P_n + nP_{n-1} = 0 \text{ (Gauss)},$$
$$P'_{n+1} - \mu P'_n = (n+1)P_n.$$

Further, it was shown that P_n was (one) solution of the differential equation:

$$(1 - \mu^2)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + n(n+1)y = 0.$$

4. Not long afterwards certain other functions called the associated Legendre functions, denoted by $P_n^m(\mu)$, were introduced; these are generalizations of the functions P_n, which correspond to the case $m = 0$. They may be defined by the formula:

$$P_n^m(\mu) = (1 - \mu^2)^{\frac{1}{2}m}\frac{d^m P_n(\mu)}{d\mu^m} = \sin^m\theta \frac{d^m P_n(\cos\theta)}{d\cos\theta^m},$$

so long as m and n are integers. The theory of these functions has been developed from a general standpoint, n not being restricted to integer values, nor μ to real values between ± 1 (corresponding to real values of θ); but these generalizations are not of much interest in the physical applications here to be considered, and I shall throughout deal only with real values of θ and positive integral values of n and m. In this case, since P_n is a polynomial of degree n in μ, it is clear from the above definition that when $m > n$, $P_n^m = 0$.

The functions P_n^m satisfy the differential equation

$$(1 - \mu^2)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + \left\{n(n+1) - \frac{m^2}{1 - \mu^2}\right\}y = 0,$$

of which Legendre's equation is a special case. Many other properties of the P_n functions are likewise special cases of properties possessed by the functions P_n^m, and, in particular, the integral properties

$$\int_{-1}^{1} P_n^m(\mu) P_r^m(\mu) d\mu = \begin{cases} 0 & \text{if } r \neq n \\ \dfrac{2}{2n+1} \dfrac{(n+m)!}{(n-m)!} & \text{if } r = n. \end{cases}$$

5. Let us now return to the general theory of the potential V in free space, which must satisfy Laplace's equation $\nabla^2 V = 0$. Any function $f(x, y, z)$ which satisfies this equation is called a harmonic function. Many such functions can be written down at sight, *e.g.* :

$$x,\ y,\ z,\ xy,\ yz,\ zx,\ xyz,\ x^2 - y^2,\ y^2 - z^2,\ z^2 - x^2,$$

while a less obvious but very important one is $\frac{1}{r}$, where $r^2 = x^2 + y^2 + z^2$. All these functions are of homogeneous degree in x, y, z ; the general solution V, of course, involves terms of all degrees.

Since differential operators are commutative, if $f(x, y, z)$ be any solution of $\nabla^2 V = 0$, $Df(x, y, z)$, where D is any space-differential operator, is also a solution : because $\nabla^2 f = 0$, and so $D\nabla^2 f = 0$, therefore $\nabla^2 Df = 0$. This gives a powerful means of generating new solutions from any known particular solution. For example :

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3} \quad \text{or} \quad -\frac{\cos\theta}{r^2} \ (\text{where } z = r\cos\theta),$$

which is the potential of a magnetic or electric doublet. This method of generating harmonic functions was developed by Maxwell in his theory of poles.

6. In discussing particular physical problems, involving solutions of Laplace's equation, certain boundary conditions have usually to be imposed on the solution : these conditions are often expressed most simply in terms of co-ordinates other than Cartesian. For example, if V has to satisfy a certain condition over the surface of a sphere, $r = a$, it is convenient to express V in terms not of x, y, z but of spherical polar coordinates r, θ, ϕ. A homogeneous harmonic function expressed in terms of these coordinates is called a spherical harmonic function. Such functions form the special subject of this lecture.

The potential function which may be regarded as the most fundamental of all, namely $\frac{1}{r}$, is obviously a very special spherical harmonic function. The general spherical harmonic function involves all three variables r, θ, ϕ. Any *homogeneous* harmonic function $f(x, y, z)$ (of degree n, say) expressed in terms of x, y, z, can at once be transformed into a spherical harmonic function by substituting

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta;$$

the resulting expression will have the form $r^n F(\theta, \phi)$; n may be negative or positive. While there are in general many harmonic functions of any assigned degree (*e.g.* six have been given for the case $n = 2$), the general homogeneous expression of degree n is, of course, not a solution of $\nabla^2 V = 0$; it is subject to certain limitations. The function $F(\theta, \phi)$ in the expression $r^n F(\theta, \phi)$ for a harmonic $f(x, y, z)$ of degree n is subject to corresponding limitations. These are expressible most readily by transforming the differential equation itself, $\nabla^2 V = 0$, in terms of new independent variables r, θ, ϕ. A simple method of doing this, not only for the coordinates r, θ, ϕ, but for any

set of orthogonal coordinates, was discovered by Kelvin, founded on a hydrodynamical interpretation of $\nabla^2 V$; in the present case $\nabla^2 V$ becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2};$$

on substituting $r^n F(\theta, \phi)$ in this expression the result is

$$r^{n-2}\left[n(n+1)F+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}\right],$$

or, in terms of $\mu(=\cos\theta)$ and ϕ,

$$r^{n-2}\left[n(n+1)F+\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial F}{\partial\mu}\right\}+\frac{1}{1-\mu^2}\frac{\partial^2 F}{\partial\phi^2}\right],$$

or

$$r^{n-2}\left[(1-\mu^2)\frac{\partial^2 F}{\partial\mu^2}-2\mu\frac{\partial F}{\partial\mu}+n(n+1)F+\frac{1}{1-\mu^2}\frac{\partial^2 F}{\partial\phi^2}\right].$$

If $r^n F(\theta, \phi)$ is a harmonic function, this must be zero; it is convenient in this case to denote $F(\theta, \phi)$ by $S_n(\theta, \phi)$; omitting the factor r^{n-2}, the equation for S_n is thus obtained as

$$(1-\mu^2)\frac{\partial^2 S_n}{\partial\mu^2}-2\mu\frac{\partial S_n}{\partial\mu}+n(n+1)S_n+\frac{1}{1-\mu^2}\frac{\partial^2 S_n}{\partial\phi^2}=0.$$

Any function S_n satisfying the equation is called a spherical *surface* harmonic of degree n, because it indicates the nature of the variation of the complete harmonic function $r^n S_n(\theta, \phi)$ over any sphere ($r=$constant).

The degree n of the harmonic only affects the differential equation for $S_n(\theta, \phi)$ in one term, $n(n+1)S_n$, arising from the term

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)$$

in $\nabla^2 V$. It is evident that if $-(n+1)$ be substituted for n, the value of $n(n+1)$ is unaltered, because

$$\{-(n+1)\}\{-(n+1)+1\}=\{(n+1)n\}=n(n+1).$$

Hence it follows that not only is $r^n S_n(\theta, \phi)$ a harmonic function, but so also is $\frac{1}{r^{n+1}}S_n(\theta, \phi)$.

Since $r^n S_n$ is supposed to be a rational homogeneous function of degree n in x, y, z combined, the equations of direct transformation show that $S_n(\theta, \phi)$ must involve $\sin\theta$ or $\cos\theta$ to degree n at most, and $\sin\phi$, $\cos\phi$ likewise. Hence, or merely by remembering that $S_n(\theta, \phi)$ must, if single valued, be periodic in ϕ with period 2π, it follows that S_n can be expanded in a finite series in ϕ of the form

$$\sum_{m=0}^{n} S_{n,m}(\theta)\cos(m\phi+\epsilon_m),$$

where the coefficients $S_{n,m}$ are naturally functions of θ. On substituting this in the differential equation satisfied by $S_n(\theta, \phi)$, we get

$$\sum_{m=0}^{n}\left[(1-\mu^2)\frac{d^2 S_{n,m}}{d\mu^2}-2\mu\frac{dS_{n,m}}{d\mu}+\left\{n(n+1)-\frac{m^2}{1-\mu^2}\right\}S_{n,m}\right]\cos(m\phi-\epsilon_m)=0.$$

Since this is to be true for all values of ϕ, each term of different frequency m in ϕ must separately vanish. Hence in S_n the corresponding factors $S_{n,m}$, as functions of μ, must satisfy the equation

$$(1-\mu^2)\frac{d^2 S_{n,m}}{d\mu^2}-2\mu\frac{dS_{n,m}}{d\mu}+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]S_{n,m}=0;$$

of the two solutions of this equation the one expressible in positive powers of μ, which alone concerns us, is the function $P_n^m(\mu)$, or a merely numerical

multiple of it by an arbitrary factor a_n^m, say. Thus the most general surface harmonic of degree n is

$$\sum_{m=0}^{n} a_n^m P_n^m(\mu) \cos(m\phi + \epsilon_m),$$

where a_n^m and ϵ_m are arbitrary. The most general *harmonic function* expressible in integral powers of x, y, z is

$$\sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} a_n^m P_n^m(\mu) \cos(m\phi + \epsilon_{n,m}),$$

while the most general harmonic function expressible in a series of integral powers of $1/r$ is

$$\sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^{n+1} \sum_{m=0}^{n} a_n^m P_n^m(\mu) \cos(m\phi + \epsilon_{n,m}).$$

7. It remains to mention the expansion-property of spherical harmonic functions. Just as any ordinary function of a single variable, defined over a range which without loss of real generality can be taken as from $-\pi$ to π, can be expanded in the form of a Fourier's series, so can any ordinary function of θ and ϕ, having specified values over the range 0 to π for θ and 0 to 2π for ϕ, be expanded in a series of surface harmonics $S_n(\theta, \phi)$. Thus

$$f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(\mu) \{A_n^m \cos m\phi + B_n^m \sin m\phi\} \equiv \sum_{n=0}^{\infty} S_n(\theta, \phi).$$

The coefficients A_n^m and B_n^m can be determined by the integral properties of the function P_n^m, which lead to the formulae

$$\left.\begin{matrix} A_n^m \\ B_n^m \end{matrix}\right\} = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_0^{2\pi} \int_{-1}^{1} f(\theta, \phi) P_n^m(\mu) \left\{\begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix}\right\} d\mu \, d\phi;$$

the integration is naturally somewhat complicated when n and m are large, unless $f(\theta, \phi)$ is a specially simple function.

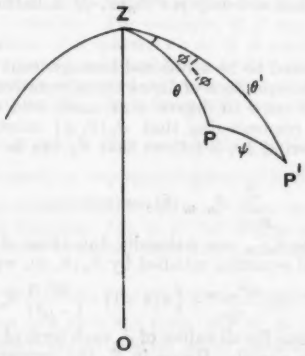

FIG. 2.

One interesting expansion of this kind is that of a Legendre function $P_n(\cos\psi)$, relative to an axis OP', in terms of surface harmonics relative to another axis OZ. By spherical trigonometry we have

$$\cos\psi = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi').$$

Then it may be shown that

$$P_n(\cos\psi)=P_n(\cos\theta)P_n(\cos\theta')+2\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi-\phi'),$$

in which every term, like $P_n(\cos\phi)$ itself, is a surface harmonic of degree n; of course the expansion of a general function usually contains harmonics of every degree.

8. Suppose now that we have a distribution of gravitating, electric and magnetic matter and that we know the potential $V=f(\theta,\phi)$ over the surface of a sphere of radius a and centre O, not passing through any of the matter; $f(\theta,\phi)$ can be expressed as a series of harmonic functions of every degree. This expansion divides $f(\theta,\phi)$ up into parts $S_n(\theta,\phi)$ whose variation with respect to r can have only two forms, proportional to r^n or $1/r^{n+1}$, or a combination of these two.

If there is no matter outside the sphere, then we know that V must vanish at infinity, and therefore no positive powers of r can occur, *i.e.* we know that, everywhere over and outside the sphere,

$$V=\sum_{n=0}^{\infty}(a/r)^{n+1}S_n(\theta,\phi);$$

if, on the contrary, all the matter is outside the sphere, and none inside, then V must be finite throughout the interior, and therefore no *negative* power of r can appear in V, which must consequently be given by

$$V=\sum_{n=0}^{\infty}(r/a)^nS_n(\theta,\phi).$$

If, however, neither of these conditions is fulfilled, so that there is matter both inside and outside the sphere, that is, the field proceeds from both without and within,

(1)
$$V=\sum_{n=0}^{\infty}\sum_{m=0}^{n}P_n^m(\cos\theta)\times\left[\left\{c_n^m\left(\frac{r}{a}\right)^n+(1-c_n^m)\left(\frac{a}{r}\right)^{n+1}\right\}A_n^m\cos m\phi+\left\{s_n^m\left(\frac{r}{a}\right)^n+(1-s_n^m)\left(\frac{a}{r}\right)^{n+1}\right\}B_n^m\sin m\phi\right],$$

where c_n^m, s_n^m are numbers lying between 0 and 1; they represent the fractions of the harmonic terms $P_n^m\cos m\phi$ or $P_n^m\sin m\phi$ in V which are due to matter *outside* the sphere.

Over the sphere itself, $r=a$, $V=\Sigma S_n(\theta,\phi)$ in all three cases; but in the present case there is an infinity of unknown numbers c_n^m, s_n^m associated with V, which are undetermined by the mere knowledge of V itself over the sphere $r=a$. They can be determined, however, if we know, besides V, its radial gradient $-\dfrac{\partial V}{\partial r}$, which is in fact the radial component of force over the sphere; for if we expand the known value, say $F(\theta,\phi)$, of $-\dfrac{\partial V}{\partial r}$, in a series like that found for $f(\theta,\phi)$, using the method already given to determine the coefficients (§ 7), we have, say,

(2)
$$-\frac{\partial V}{\partial r}=\frac{1}{a}\sum_{n=0}^{\infty}\sum_{m=0}^{n}P_n^m(\cos\theta)\{a_n^m\cos m\phi-\beta_n^m\sin m\phi\}\text{ over }r=a,$$

which may be compared with the expression obtained by differentiating the above formula for V with respect to r, *i.e.*

$$(3)\quad -\frac{\partial V}{\partial r}=\frac{1}{a}\sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta)[\{nc_n^m-(n+1)(1-c_n^m)\}A_n^m\cos m\phi + \{ns_n^m-(n+1)(1-s_n^m)\}B_n^m\sin m\phi],$$

where r has been set equal to a after differentiation; the coefficients of each separate harmonic term $P_n^m\cos m\phi$ or $P_n^m\sin m\phi$ for each n and m must be equal in the two expressions, giving

$$(4)\quad \alpha_n^m=\{nc_n^m-(n+1)(1-c_n^m)\}A_n^m,$$

$$(5)\quad \beta_n^m=\{ns_n^m-(n+1)(1-s_n^m)\}B_n^m,$$

from which, since α, β, A, B are known, c_n^m and s_n^m can be determined. From a knowledge of V and its radial gradient over the surface of a single sphere, we have thus ascertained exactly how much of each separate harmonic term in V is due to inside and how much to outside causes.

9. As has been seen, spherical harmonic functions were introduced in the course of researches on purely dynamical and gravitational problems, such as those of the Earth's figure, and the tides. Their first application to mathematical physics was made by Gauss in 1839, in a great memoir on the Earth's magnetic field. Gilbert, in 1600, by means of experiments on spherical pieces of loadstone, had reached the conclusion that the Earth's magnetic field proceeds from within, and is substantially that of a uniformly magnetized sphere, though with many minor irregularities. Between his time and that of Gauss, the direction and intensity of the magnetic force had been measured at many points scattered widely over the Earth's surface, and represented on maps by isomagnetic lines, or lines along each of which a particular magnetic component, or angle of direction, has an assigned value (analogous to the contour lines by which the elevation of the land is indicated on ordinary maps). No progress had, however, been made by any one before Gauss in representing the field in a mathematical form. Gauss assumed, what is very nearly correct, that there is no magnetic matter or electric current in the atmosphere near the ground, so that the magnetic field there must possess a potential function V satisfying Laplace's equation and representable by a series of spherical harmonics as in (1).

This potential cannot be measured directly, however; what is measured are the three components of force, H horizontal and northwards $\left(=\frac{\partial V}{r\partial\theta}\right)$, W horizontal and westwards $\left(=\frac{\partial V}{r\sin\theta\partial\phi}\right)$, and Z vertical and downwards $\left(=\frac{\partial V}{\partial r}\right)$, at the Earth's surface, which is $r=a$, if a now denotes the Earth's radius. Thus we know, or by analysis of the actual observations of H, W, Z can find, to a degree of completeness and accuracy conditioned by the observations, the coefficients α_n^m and β_n^m in (2) and, effectively, the coefficients in the two following series derived from (1)

$$(6)\quad (W)_{r=a}=\left(\frac{\partial V}{r\sin\theta\partial\phi}\right)_{r=a}=\frac{1}{a\sin\theta}\Sigma\Sigma P_n^m\{-mA_n^m\sin m\phi+mB_n^m\cos m\phi\},$$

$$(7)\quad (H)_{r=a}=\left(\frac{\partial V}{r\partial\theta}\right)_{r=a}=\frac{1}{a}\Sigma\Sigma\frac{dP_n^m}{d\theta}\{A_n^m\cos m\phi+B_n^m\sin m\phi\}.$$

The coefficients in (6) are found by expanding $aW\sin\theta$ in the form found for $f(\theta,\phi)$ in § 7; those in (7) are less easily found from the observations of H, because the expansion is not directly in a series of surface harmonics, but of functions $\frac{dP_n^m}{d\theta}$, which, however, may themselves be expressed in harmonic form.

It will be noted that (6) and (7) both contain A_n^m and B_n^m, and, as Gauss pointed out, if we know W all over the Earth, the value of H can be deduced from it, by substituting for A and B from (6) in (7); if the results did not agree, it would show that the field was not derivable completely from a potential V, *i.e.* that the atmosphere must either be magnetic or convey electric currents. Gauss showed that in fact, to the accuracy permitted by the observations of H and W available in his time, the values of H were consistent with the A's and B's derived from W. These indicate the form of V itself (apart from an arbitrary constant), so that the data for $\frac{\partial V}{\partial r}$ can be used to determine whether the Earth's field is of internal or external origin. Gauss showed that, as nearly as his data for Z could indicate, they agreed with Gilbert's conclusion, *i.e.* V is of the form $\sum_{n=0}^{\infty} (a/r)^{n+1} S_n(\theta, \phi)$, or $c_n^m = 1$ for every n and m; actually by far the most important term in V is the first, $\frac{aP_1(\cos\theta)}{r^2}$ or $\frac{a_1 \cos\theta}{r^2}$, which corresponds to the field of a uniformly magnetized sphere or, what is equivalent at outside points, that of an infinitesimal magnet at the Earth's centre, O, directed along the axis of rotation.

The next most important is a term containing $P_1^1 \cos(m\phi + \epsilon)$, which represents the field of an elementary magnet at O lying in the plane of the equator. These two terms can be combined into one (by using the last formula of § 7, putting $n = 1$), namely, $P_1(\cos\psi)$ about an axis OP', inclined at about 10° to the Earth's axis of rotation; the ends of this axis are called the poles of magnetization of the Earth, the northern one being situated to the north of Canada. Owing to the presence of other, smaller, terms in V, the horizontal magnetic force does not vanish at these places, but at other points somewhat removed from them, and it is these points, where the magnetic needle becomes vertical, that are usually called the magnetic poles.

This investigation by Gauss not only gave mathematical form and precision to Gilbert's conclusion, but also enabled the larger irregularities in the Earth's field to be expressed analytically, and extended to these irregularities the conclusion, which Gilbert's methods could scarcely have established, that the field is of internal origin.

Subsequent analyses with better data have enabled more terms in V to be calculated, and have suggested that a small part—about 3 per cent.—of the Earth's surface magnetic force is of external, presumably upper atmospheric, origin, while another small part, of similar amount, may not be derivable from a potential (*i.e.* the above expressions for H and W do not give quite concordant A's and B's), and is therefore due to currents traversing the lower atmosphere.

Adams, the astronomer, spent much of the later part of his life in calculating V to a great many terms, more, indeed, than the accuracy and completeness of his data would warrant. The more terms are taken, the greater the difficulty of the computations, and in practice it is not worth while going to more than about ten terms, until better and much more detailed observations are taken than we now have.

This, the first application of spherical harmonic analysis to mathematical physics, is also one of the most simple, interesting and successful. It effects a definite separation of the parts of the Earth's magnetic field, due to internal and external causes; moreover, if we neglect the very small external part, and assume the space above the Earth to be wholly free from electric currents, the analysis enables the nature of the magnetic field to be inferred at any point outside. It shows that the further we go outwards, the less important do the harmonics of higher degree become relative to the main harmonic representing the field of a uniformly (obliquely) magnetized sphere; and this part of the

field varies as $\frac{1}{r^3}$, so that, though it decreases only slowly, it does so more quickly than the Earth's *gravitational* force.

We cannot extend this conclusion to the Earth's interior, however; we know that the field originates there, but spherical harmonic analysis throws no light on how, or exactly where, it is produced, whether by magnetic deposits or by electric currents. This, indeed, remains an unsolved problem, probably to be overcome, if at all, only by further knowledge of physics.

10. After Gauss, the first to make a new application of spherical harmonic analysis to the Earth's magnetism was Schuster, who, in 1889, fifty years later, considered the daily variation of the Earth's magnetic field. The magnetic components, such as H (or W or Z), at any place on the Earth may be approximately expressed in the form

$$\overline{H} + H(t),$$

where $\overline{H}$ denotes the mean value, and $H(t)$ the variation from the mean at the time t (Greenwich time, say). Naturally $\overline{H}$ and $H(t)$ are functions of θ and ϕ. The analysis made by Gauss referred to the mean values of the magnetic components, $\overline{H}$, $\overline{W}$, $\overline{Z}$, whereas Schuster considered $H(t)$, $W(t)$ and $Z(t)$. He analyzed these by the method of Fourier in the form

$$H(t) = \sum_{p=1}^{\infty} \left(h_c^p \cos \frac{2\pi pt}{T} + h_s^p \sin \frac{2\pi pt}{T} \right)$$

appropriate to a function periodic in a day (T); actually he used only the terms up to $p=4$, the coefficients h_c^p, h_s^p (and likewise those for W and Z) being functions of θ and ϕ.

Assuming that the daily-varying part of the field is due to a potential $V(t)$, a periodic function of time, this also must be expressible as a Fourier's series:

$$V(t) = \sum_{p=1}^{\infty} \left(V_c^p \cos \frac{2\pi pt}{T} + V_s^p \sin \frac{2\pi pt}{T} \right),$$

the coefficients V_c^p, V_s^p being functions of r, θ and ϕ. By differentiation of V_c^p or V_s^p with respect to r, θ or ϕ we must be able to obtain h_c^p, w_c^p, z_c^p or h_s^p, w_s^p, z_s^p. Actually, as in Gauss's analysis, we know the latter but not the former. Proceeding as in that case, however, we can determine V_c^p, V_s^p by analyzing z_c^p, z_s^p and $w_c^p \sin\theta$, $w_s^p \sin\theta$ into series of spherical harmonic functions, and deducing the internal and external parts of the field; also we can then deduce h_c^p, h_s^p and compare with the observed values, to check whether the field really is derivable from a potential function V satisfying Laplace's equation.

This analysis, which is similar, in principle, to that made by Gauss, though more complicated in its details, gave results differing from his in an interesting way. It was found sufficient to deal only with one or two principal harmonics in each V_c^p or V_s^p; these harmonics proved to be P_p^p or P_{p+1}^p. The four periodic terms $p=1$ to $p=4$ thus involve eight harmonics in all; each of these proves to have a part produced above the Earth and a part of internal origin, these being in the ratio about 5 to 2. Thus whereas in the case of $\overline{H}$, etc., discussed by Gauss, the internal part was the greater, here the reverse is true.

At different seasons of the year, and in different years, $V(t)$ and its harmonic components have different values; the components do not all increase or decrease in the same ratio—their ratios vary among themselves. But throughout these changes the ratio between the external and internal parts of each harmonic remains constant, and this argues a connection between the two parts. As Schuster perceived, this connection almost certainly is a

directly causal one: the internal part of the field is due to electric currents induced within the Earth by the varying outer magnetic field, regarded as primary.

If there is any electrically conducting material within the earth, such currents must certainly be induced; but it is necessary to examine mathematically whether their field would bear to the outer, inducing field the same connection as that found between the inner and outer parts of the Earth's daily-varying field. This question was settled in the affirmative by Lamb, in a series of researches in which spherical harmonic functions were used to investigate the oscillations of air inside a sphere, or the oscillations of an elastic solid sphere, or the induction of electric currents within a conducting sphere. The analysis is very similar in all three cases, which can be thus treated because in each there occurs a potential of some kind, satisfying an equation involving $\nabla^2 V$.

In the present case, the inner and outer parts of the daily-varying field are consistent with the hypothesis of induction, and indicate that the main conducting material does not extend above a depth of about 200 miles below the Earth's surface: moreover, while it is far more conducting than dry earth or rock, it is only about one-hundredth as conducting as sea water. The study of the daily magnetic variations thus gives us information, about inaccessible regions of the Earth's interior, which can hardly be obtained in any other way.

The origin of the main, external portion of the daily-varying part of the Earth's field constitutes a problem which has not yet been definitely solved, though it is probably nearer solution than is the origin of the Earth's main field. Its cause is generally agreed to be in the high regions of the atmosphere where the air is ionized—the layer often called after Heaviside. According to one theory, suggested by Balfour Stewart and developed mathematically by Schuster, electric currents are induced in this layer by a large-scale convective circulation of the air in the presence of the Earth's main field. I have called this the dynamo theory, because the air here plays the part of a moving armature in the field of the great magnet of the Earth. It is doubtful, however, whether the currents thus produced, and which must certainly exist, are sufficiently intense to explain the observed daily magnetic variation.

11. Another theory, proposed only last year, may be mentioned because of its mathematical interest, though I do not credit it as a solution of the problem under consideration. In the very high atmosphere the free electrical charges, electrons or ions, will be unable to describe straight paths except along the Earth's lines of force: if their motion is at all oblique to these, the field causes the charges to describe spirals having their axes directed along the magnetic force. Positive charges spiral in one direction, negative in the opposite; in either case they behave as a tiny circular current, plus a current along the line of the Earth's field. The latter part of the current, when summed over all the charges, is zero, because as many travel in one direction as the other; but the circular motion produces currents which are the same in direction for all the particles, positive or negative; and these little circular currents are equivalent to, and in fact *are*, tiny electromagnets, directed *opposite* to the field. Thus the high atmosphere is effectively magnetized opposite to the field; the intensity of its magnetization is proportional solely to the number of free charges in each part of the layer.

Now the number of charges is much more by day than by night, being governed largely by the sunlight falling on the atmosphere. Thus the high magnetic layer of air is stronger over the sunlit day hemisphere than over the night hemisphere, and though we have at present no accurate knowledge of its actual strength at any point, the distribution of its *relative* strength can be guessed with fair probability, while its direction of magnetization at each point is, of course, known. The author of this hypothesis attempted to determine the potential of the magnetic field of this layer by a method which is

equivalent to the calculation of the potential of a gravitational distribution from the formula $\int \frac{dm}{r}$, though in the case of a magnetic distribution the calculation is of a still higher order of difficulty; when it is recalled that by this method a separate calculation of V has to be made for each point P at which V is to be determined, and that each such calculation involves the calculation of the distance r from P to every other element of the spherical magnetized shell, the numerical laboriousness, difficulty and unsuitability of the method—if any other is available—become clear.

An alternative method is afforded by spherical harmonic analysis. The radial and tangential components of the magnetization of the spherical shell have to be treated separately, but, roughly speaking, when their assumed distributions have been expressed in the form of a series of surface harmonics $\Sigma a_n S_n(\theta, \phi)$—and this is a relatively simple matter—the two corresponding parts of the potential V can be written down at once, and together give the potential for every point whatsoever. It may be of interest to indicate the nature of the integrations involved in the determination of the coefficients A_n^m, B_n^m of the harmonic expansions in this case:

$$\int_{-1}^{1} \frac{d\mu}{\sqrt{1+3\mu^2}}, \quad \int_{-1}^{1} \frac{d\mu}{(1+3\mu^2)^2}.$$

When these two integrals have been evaluated—not a difficult matter—all the necessary coefficients A_n^m, B_n^m can be determined by recurrence-formulae based on those for the functions P_n, and the formula for P_n^m in terms of P_n, given earlier in this lecture. [A slide was shown indicating how successive terms of the harmonic series contribute to the total value of various functions involved in the physical discussion: in each case four terms give sufficient accuracy.]

12. Thus far my illustrations of the application of spherical harmonic functions in mathematical physics have been taken from the field of magnetism, and especially the magnetization of the great sphere which we inhabit. Many other electric and magnetic problems relating to spheres can be found in treatises on electricity, such as those of Maxwell or Jeans, while for hydrodynamical applications reference may be made to Lamb's treatise.

For my last illustration I will turn to the latest development of modern physics, namely, Schrödinger's wave-mechanics. The few remarks on this subject, which are all that my remaining time will allow, are taken from a very interesting brief account of wave-mechanics which is given, almost incidentally, in a recent book called *Polar Molecules* by P. Debye.*

It has long been recognized that of the two branches of optics, namely, geometrical optics and physical wave-optics, the former can be considered as a special case of the latter, corresponding to vanishingly small wave-length of the light in reference to any diffracting objects or inequalities of the medium. In geometrical optics light is regarded as travelling in straight lines in a uniform medium, though in a non-uniform medium the rays are bent: the procedure followed is to trace the course of the rays. In wave-optics, on the other hand, the waves are followed in detail, and it is found that, for example, there is a bright spot in the centre of the shadow of a circular disc, thrown by a point source of light on the axis of the disc. In the wave theory we deal with the fundamental equation:

$$\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0,$$

where ψ may, for example, represent a component of the electric intensity, and v is the velocity of propagation, which will vary from point to point in a non-uniform medium.

* Chapters 8 and 9.

Now let us turn to classical mechanics. Here it has been customary to start by considering the motion of a point particle, and until recently this proved sufficient. But in dealing with the mechanics of the smallest particles, atoms, molecules, and electrons, difficulties arose, which have gradually shown themselves to have certain analogies with the insufficiency of geometrical optics to describe dispersion, interference and diffraction of light, properties which depend essentially on the fact that light is a wave-phenomenon, and that each band of light has its own wave-length. A French physicist, de Broglie, first suggested that material particles and electrons are likewise wave-phenomena: this idea was subsequently developed and mathematically formulated by Schrödinger, while recently Thomson, and Davisson and Germer, have shown that electrons can actually be diffracted by metal films, like X-rays, and have determined their wave-lengths—depending on their speeds.

Schrödinger's wave-equation for a particle of mass m, total energy W, and potential energy V is

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}(W - V)\psi = 0,$$

where h is Planck's quantum constant. It would take too long to tell in detail how this equation is arrived at. The function V is supposed known, depending on the nature of the field in which the particle moves. For example, for a hydrogen atom, in which we deal with the electron, $V = e^2/r$. The nature of the function ψ is a matter as to which there are divergent interpretations, but notwithstanding this the results produced from this equation, by a mixture of mathematical analysis and physical guessing or interpretation, are marvellous.

In the first place, the requirement that ψ must be everywhere finite and single-valued is found to limit W to a discrete series of special values, the same, in the case of the hydrogen (H) atom, as those found by Bohr.

Another case, described by Debye, is that of a rotating molecule—such as HCl, the H and Cl atoms being regarded as point masses, a apart, and as equivalent to an electric doublet—in an electric field. Taking the distance between the nuclei as constant, the independent coordinates and variables are θ and ϕ, θ being measured from the direction of the field E. In this case the equation becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\theta^2} + \frac{8\pi^2 m a^2}{h^2}(W + \mu E\cos\theta)\psi = 0.$$

In the special case $E = 0$, when there is no potential energy, the only finite single-valued solutions of this occur when

$$W = n(n+1)\frac{h^2}{8\pi m a^2},$$

and then

$$\psi = P_n^p \cos(p\phi + \epsilon).$$

This solution is also the first approximation to the solution when E, though small, is not zero. The results are of importance in the theory of the dielectric constant or specific inductive capacity of gases, including the variation with temperature.

The theory also bears on the absorption and dispersion of light by such a gas. Following the analogy afforded by Bohr's theory of spectra, it is supposed that the gas emits and absorbs radiation of frequency ν given by

$$h\nu_{12} = W_1 - W_2$$

when the energy of a molecule changes from W_1 to W_2. This limits the possible frequencies of light emission, but says nothing as to the intensity of the emission. In Bohr's theory this was inferred from a rather unsatisfactory new principle called the correspondence principle. Schrödinger, however,

derives the intensity as well as the frequency (or the energy-levels W) from his one equation, by treating the product $\psi_1\psi_2$ as a density of electricity in the system, this density oscillating with the frequency ν_{12}. The total oscillating electric moment of the system is a directed quantity depending on the integral of $\psi_1\psi_2$ (multiplied by certain factors) throughout space. In calculating such integrals, great use is made of the integral properties of the P_n^m functions, already referred to. The results prove to be zero in general, for an HCl molecule not in an electric field; thus only a few special frequencies, ν_{12}, out of the ∞^2 set of values afforded by the W's, are found to be effective. In this and the other cases in which Schrödinger's equation for particular systems have been solved, the results are in remarkable agreement with observation.

These illustrations, cursory as they are, will, I hope, have given some slight idea as to the ways in which spherical harmonic functions prove useful in mathematical physics. It may be said that they occur wherever an equation is involved in which ∇^2 appears, and in which spherical polar coordinates are appropriate. Any attempt to dispense with them in such problems is likely to prove disastrous or, at least, to add enormously to the difficulty of the discussion. On the other hand, it unfortunately proves very laborious to deal even with spherical harmonic functions when many terms need to be taken. For example, the ocean tides on the Earth can only be completely discussed when full account is taken of the form of the land boundaries and ocean beds; but no one has yet attempted to do this, because it would involve expressing the ocean depth as a series of spherical harmonics. But to say that in very difficult problems the use of spherical harmonic functions is laborious is not to slight the method, because any other would be still more difficult.

S. CHAPMAN.

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780. J. R. Young, Professor at Belfast—"a place of many religious sores . . . was unequally used in reference to his retiring pension: but this matter was afterwards righted by the Government. . . . We remember this by the opposition made by some honourable member who rather suspected that a Belfast payment must be a job. The following dialogue ensued:

Hon. Member. . Is Mr. Young a professor of theology?

Current Minister. No! of mathematics.

Hon. Member. I withdraw my opposition.—*Athenaeum*, 1859, i. p. 321.

781. Flamsteed spelt the name of his friend Towneley in every way in which the letters would sound the word: and Rigaud adopted Townley on the authority of the *Philosophical Transactions*. . . . The name of Horrocks still exists in Lancashire. . . . *It was thus that Horrocks spelt his own name.* In the work which once belonged to him, which was picked up by Professor De Morgan, and deposited in Trinity College Library, it was so spelt. Until somebody shall be proved to know better than Horrocks himself how to spell his name, we intend to follow him.—From a review of a *Memoir of the Life of the Rev. Jeremiah Horrox, Ath.* 1859, ii. 295 [probably written by De Morgan].

782. Should any one possess an astronomical work, containing handwriting, and having, with a proper date, the Towneley book-plate, the chances are considerable that it belonged to Horrocks. The coat-of-arms must be described to the uninitiated in heraldry as three black stars over a black bar, with a falcon for a crest. The matter will be almost certain if the book be one of those which are . . . in the *Companion to the Almanac* for 1837.—*Loc cit.*

783. A destroyer action against battleships, with both sides calculating their hardest, is much better described as an attack by massed logarithms upon a set of quadratic equations.—J. M. N. Jeffries, *Daily Mail*, 26th March, 1930.

MATHEMATICAL NOTES.

968. [L¹. 19. a.] *A generalisation of Feuerbach's Theorem.*

In a recent number of the *Gazette* (May 1930, p. 111), Prof. L. J. Rogers proved the following extension of Feuerbach's Theorem:

"If a conic touches the sides of a triangle ABC, the locus of the intersection of a tangent to it and of the perpendicular tangents to a confocal conic through the circumcentre O is a circle j touching the circle through the middle points D, E, F of the sides of the triangle."

The following proof may be of interest, as involving only elementary methods.

Let $\varpi_D, \varpi_E, \varpi_F$ be the powers of D, E, F for the circle j. The power ϖ_D is equal to the product of the perpendiculars from D to the two tangents of the confocal through O which are perpendicular to BC, that is, is equal to the product of the perpendiculars from O to the same two tangents of the confocal. Let the tangent at O to the confocal cut these parallel tangents in T and T'. Then, if the conic is an ellipse, $TO.OT'$ is equal to the square of the semi-diameter parallel to TOT', whatever the direction of the parallel tangents may be. If the conic is a hyperbola, it is the product $OT.OT'$ that is positive; and this is equal to the square of the corresponding semi-diameter of the conjugate hyperbola. But in either case $OT.OT'$ has a value κ that is independent of the direction of the parallel tangents; and therefore, if λ, μ, ν are the angles which the tangent at O makes with BC, CA, AB, then

$$\varpi_D = \kappa \cos^2\lambda, \quad \varpi_E = \kappa \cos^2\mu, \quad \varpi_F = \kappa \cos^2\nu,$$

and all the powers have the same sign.

Now let L be a limiting point of the circles j and DEF. We want to show that L lies on the circle DEF, that is, that

$$DL.EF \pm EL.FD \pm FL.DE = 0.$$

It is well known that $\varpi_D : \varpi_E : \varpi_F = DL^2 : EL^2 : FL^2$.

Hence, in order to prove that the circles j and DEF touch one another, we must show that either

$$\sin A\sqrt{\varpi_D} \pm \sin B\sqrt{\varpi_E} \pm \sin C\sqrt{\varpi_F} = 0,$$

or
$$\sin A\sqrt{(-\varpi_D)} \pm \sin B\sqrt{(-\varpi_E)} \pm \sin C\sqrt{(-\varpi_F)} = 0.$$

But this is obvious, since $\sin A = \pm \sin(\mu - \nu)$, etc., and

$$\varpi_D : \varpi_E : \varpi_F = \cos^2\lambda : \cos^2\mu : \cos^2\nu.$$

Actually the first relation holds if the confocal through O is a hyperbola, the second if it is an ellipse.

H. HILTON.
E. H. NEVILLE.

969. [A. 1. a; V. 7.] *A Letter of De Moivre and a Theorem of Halley. V.* Note 947, *Math. Gaz.* vol. xiv, p. 574.

The following extract from Maseres's *Scriptores Logarithmici*, vol. 5 (1804), p. 519, explains itself:

"My learned friend Mr. Nicholas Vilant, Professor of Mathematicks in the University of St. Andrew's in Scotland, has informed me that he has met with a very curious Letter of the celebrated Algebräist, Mr. Abraham De Moivre, to Dr. Edmund Halley, concerning one of the abstruse Algebräick expressions given by the latter in his *Discourse on Compound Interest*. . . . He met with this Letter many years ago in a periodical publication that came out in the year 1761, intitled *The Mathematical Magazine, and Philosophical Repository*, which was undertaken by Mr. G. Witchell, and Mr. T. Moss, and some other persons, and printed for J. Wilkie at the Bible in Saint Paul's Church-yard. This publication, he informs me, was a very useful one: the part of it called *The Mathematical Magazine* contained many curious papers taken from foreign Literary Memoirs and Journals, or furnished by learned foreign correspondents;

and *The Philosophical Repository* promised to be a most valuable work, and was to have comprehended a new and universal Dictionary of pure and mixt Mathematicks. But, not meeting with encouragement, it was soon discontinued, only five numbers of it having been published; in one of which, in pages 41, 42, and 43, the said Letter of Mr. De Moivre is to be found. Mr. Witchell, the principal undertaker of this publication, had originally been a journey-man watch-maker, but, notwithstanding the want of a learned education, had (like many other men who have become eminent in science,) attained a great degree of skill and knowledge in the Mathematicks, and was much patronized by the learned Dr. Bevis (a literary gentleman of great note at that time,) who probably communicated to him the said Letter of Mr. De Moivre to be published in his Magazine, where it was printed (as it is supposed,) for the first, and, perhaps, the only, time: for Mr. Vilant never met with it in any other book. The other undertaker of that useful Magazine, Mr. T. Moss, was (as Mr. Vilant informs me,) an Excise-officer, and had been a writer on the subject of *Gauging* with a deserved reputation. Mr. Witchell was considered as a person of such respectable conduct and abilities that, in the latter part of his life, he was appointed Head Master of the Royal Mathematical Academy at Portsmouth. From this Magazine Professor *Vilant* has been so kind as to copy this Letter, and to send me the Copy of it, to be published on the present occasion. The Letter itself has no date to it, but was probably written soon after the first publication of Dr. Halley's Discourse on Compound Interest, (which was in the year 1705, in the Introduction to the first edition of Sherwin's Tables of Logarithms,) when that Discourse, from the accuracy of the abstruse and undemonstrated Algebräick expressions, given in it, of the values of the rate of Interest of Money in those Problems in which that rate was the unknown quantity, must naturally have attracted the attention of Mathematicians in general, and particularly of so excellent an Algebräist as Mr. De Moivre, and have excited them to endeavour to find out the investigation of them."

Maseres then (pp. 520-522) reprints De Moivre's letter. He further remarks that De Moivre has used a different notation from that of Halley, and so (pp. 520-524) gives the letter over again, "expressed in Dr. Halley's Notation."

On pages 219-230 of the same volume Maseres reprints Halley's Discourse on Compound Interest, which gave rise to the letter. As he says, it was first published in 1705 * in the introduction to Henry Sherwin's *Mathematical Tables*; Maseres's reprint is "from the third Edition of those Tables, published by *Mr. William Gardiner* in the Year 1741." He follows it up with an elaborate series of "Notes on some difficult passages of the foregoing discourse" (pp. 231-516), in which he gives various methods for obtaining the approximations stated by Halley without proof. The particular problem with which De Moivre deals is stated thus by Halley † (Maseres, *loc. cit.* p. 226, Sherwin (1742), p. 20):

The annuity (a), *present value* (z), *and time* (t), *being given, to find* (r) *the rate of interest.*

He goes on: "This Problem being more difficult than appears at first sight, and requiring the solution of this Equation

$$\frac{a}{z} = \frac{z+a}{z} r^t - \frac{t+1}{r},$$

to which it is reduced; there must be applied some method of approaching the root r, which is by no means evident: And that approximation as the number of years and rate are greater or less, cannot properly be obtained by one general rule; but rather by two, according as the value of the reversion is greater or less.

* For the vexed question of the dates of the various editions of Sherwin's Tables, see Glaisher *Report on Mathematical Tables* (British Association, 1873), pp. 129-131.

† De Moivre puts $z/a = s$, the number of years' purchase, and writes n for t and x for r.

"If the number of years be great (as suppose 40 or upwards) and especially if the rate of interest be high $1+\frac{a}{z}$ will be nearly the rate, or more accurately $\frac{z+a}{z}-\frac{z^t}{z+a}\times\frac{a}{z}$; call it r; and $\frac{a}{r^t\times r-1}$ will be exceeding near the value of the reversion, which let be x; then $1+\frac{a}{z+x}$ shall approach the true rate sufficiently. But if greater exactness be desired, by repeating this process it will be obtained. . . .

"If the number of years be small, the aforesaid Rule will avail little. In this case it will be requisite to approach the rate thus. Let $\frac{t+1}{2}$ be the Index of a root of $\frac{at}{z}$; from which root subtract 1, and the remainder call y, and let $\frac{6}{t-1}$ be called b. I say, that $b-\overline{bb-2by}^{\frac{1}{2}}$ is sufficiently near to $r-1$, and will be still nearer the truth, as the number of years is smaller; and that the error will be always in excess."

Following De Moivre's letter Maseres (*loc. cit.* pp. 530-536) gives "A Conjecture concerning the Manner in which Dr. Edmund Halley investigated" the above approximation, adding "he himself thought fit to conceal his investigations of them, and even the foregoing Letter of Mr. De Moivre, which was found amongst Dr. Halley's papers after his death, and was published for the first time, 19 years after it, in the year 1761."

For the sake of completeness, I add a few notes upon the persons mentioned.

Nicholas Vilant succeeded David Gregory in 1765 as Professor of Mathematics in the University of St. Andrews, having matriculated there in 1753 (see *The Matriculation Roll of the University of St. Andrews* 1747-1897, ed. J. M. Anderson, 1905). He was the author of "The Elements of Mathematical Analysis, abridged, with notes demonstrative and explanatory, and a synopsis of Book V of Euclid" (Edinburgh, 1798). He died 25th May, 1807.

The following account of George Witchell, F.R.S. (1728-1785), is given by Hutton (*Philosophical and Mathematical Dictionary*, 2 vol., 1815). "He was maternally descended from the celebrated clock and watch maker Daniel Quare, in which business he was himself brought up, and was educated in the principles of the Quakers, all his progenitors for many generations having been of that community, and whose simplicity of manners and integrity of character he practised through life. It appears that Mr. W. cultivated the study of astronomy at a very early age indeed, as he had a communication on that subject published in the *Gentleman's Diary* for 1741, which must have been written before he was 13 years of age. Soon after this he became a pretty constant writer in both the *Diaries* and the *Gentleman's Magazine*, a practice which he continued a long time, sometimes under his own name, but more frequently with the initials G.W. only. In 1764 Mr. W. published a map, exhibiting the passage of the moon's shadow over England in the great solar eclipse of April 1 that year; the exact correspondence of which to the observations gained him great reputation. In the following year he presented to the commissioners of longitude a plan for calculating the effects of refraction and parallax, on the moon's distance from the sun or a star, to facilitate the discovery of the longitude at sea. Having been elected F.R.S. and taught mathematics in London * for many years with much reputation, he was, in 1767, appointed head master of the Royal Naval Academy at Portsmouth,

* A notice on the cover to Number IV of the *Mathematical Magazine* states that "G. Witchell, Teacher of the Mathematicks, . . . still continues to instruct Gentlemen, at their own Houses, in any branch of the Mathematical Sciences." From the cover to Number V it appears that he lived at "No. 6. in Union-Court, opposite St. Andrew's Church, Holbourn" (now Ely Place).

on the recession of Mr. Robertson; where he died by a paralytic stroke in 1785, at 57 years of age."

Witchell contributed to the *Philosophical Transactions of the Royal Society* (i) "A general investigation of the nature of the curve, formed by the shadow of a prolate spheroid, on a plane standing at right angles to the axis of the shadow" (*Phil. Trans.* 57 (1767), 28-43; *Abridgement* (1809), vol. 12, pp. 372-379), by which he hoped to explain irregularities in the eclipses of Jupiter's satellites (on this paper see Cantor, *Geschichte der Math.* vol. 4, p. 514), and (ii) "Extract from a letter inclosing some account of a solar eclipse observed at George's Island, by Captain Wallis, and several astronomical observations made at Portsmouth" (*Phil. Trans.* 62 (1772), 33-41; *Abridgment* (1809), vol. 13, pp. 276-7). For Samuel Wallis (1728-1795) see *D.N.B.*; his ship was the *Dolphin*; George's Island is Tahiti, in the Society Islands, then first discovered (see *Gonzalez' Voyage to Easter Island*, 1770-71, Hakluyt Society (2), 13 (1908), 67-8, 72); the eclipse was on July 25, 1767.

Thomas Moss, of Bromley, Middlesex, was the author of *Treatise of Gauging; or, the modern practical Gauger*, 2nd edition, enlarged 1768, 3rd edition 1779.*

For John Bevis or Bevans (1693-1771) see *D.N.B.* or Hutton (*loc. cit.*). He was of Christ Church, Oxford, had an observatory at Stoke Newington, compiled *Uranographia Britannica*, 1745-50, and was Fellow and Foreign Secretary, Royal Society. He was a contributor to the *Mathematical Magazine*, 1761.

F. PURYER WHITE.

Halley's Rule—referred to and proved by De Moivre in the letter quoted in Note 947—was given in a paper by Halley entitled "Of Compound Interest," published in *Sherwin's Mathematical Tables*, printed for W. and J. Mount, 1761. The paper is reprinted, with this reference, in the *Journal of the Institute of Actuaries*, vol. ix, pp. 259-269, under date April, 1861. Two points are worthy of note. I. Halley's result is $1+b-\sqrt{bb-2by}$: De Moivre gives $1+b+\sqrt{bb-2by}$, *i.e.* he has taken the wrong root of his quadratic equation. II. What Halley and De Moivre call the "rate of interest" is 1 + the rate of interest in modern terminology. In Halley's example he finds "$r=1{\cdot}068327$," the rate sought. The rate r thus found is "always some small matter too big, the true rate being 1·06814." This means that the rate of interest, in the modern sense of the term, is 6·814%. Halley's use of the term is, however, not very consistent: in another problem he brings out $r-1=0{\cdot}05096$, adding rather naïvely: "So that the rate of interest sought is 5 per cent. per annum"!

It is not evident why De Moivre should send to Halley a demonstration of his own Rule: for the Rule was evidently not a merely empirical one, and Halley must have worked it out for himself.

G. J. LIDSTONE.

970. [D. 2. c_1.] *Stirling's Theorem.*

The ingenious attempt made in Note 951 (*Gazette*, vol. xv, p. 22) to derive Stirling's Theorem from Wallis's expression for $\pi/2$ affords interesting matter for comment. It might be objected offhand that in the proof the limit a is assumed in a somewhat casual manner, and the writer of Note 966 (p. 167) exposes the weakness of the deduction dramatically by using a parallel argument to prove a paradox, but it seems worth while to probe the question a little more deeply. Freed from doubtful expedients by the way, the course of the argument is practically this:

Let $$P_n \equiv 4^n (n!)^2/(2n!)\sqrt{2n}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

Then $P_n \to \sqrt{\pi/2}$ by Wallis's result.

* I have not been able to see this; I take the entries from booksellers' catalogues.

1. LEMMA. Consider the value of $(2n!)/n!$.

$$(2n!/n!) \equiv (n+1)(n+2)\dots(n+n)$$
$$\equiv n^n(1+1/n)(1+2/n)\dots(1+n/n)$$
$$\equiv n^n u_n, \text{ say.}$$

Changing n into $n+1$ in this identity and dividing, we easily find

$$\frac{u_{n+1}}{u_n} \equiv 4\frac{2n+1}{2n+2}\left(1+\frac{1}{n}\right)^{-n}. \quad \dots\dots(b)$$

Since the limit of the right-hand side is $4/e$, as n tends to infinity, we may put $u_{n+1}/u_n = (4/e)a_{n+1}$; where we are justified in assuming that $a_{n+1} \to 1$.

Then it follows that $u_n = (u_1/1)(u_2/u_1)(u_3/u_2)\dots(u_n/u_{n-1})$

$$= (4/e)^n a_1 a_2 a_3 \dots a_n$$
$$= (4/e)^n A_n, \text{ say, for brevity.}$$

$\therefore (2n!)/n! = (4n/e)^n A_n$.

[*But it does not follow that, because* $a_n \to 1$, *the product* $a_1 a_2 \dots a_n$, *symbolised by* A_n, *tends to any finite limit at all.*]

2. *Practically as in the note*, though not in the same precise order, the concluding steps, with the amended value of $(2n!)/n!$, work out to the following results:

Changing n into $2n$ in (a) we have $P_{2n} \equiv (n!)(2n!/n!)(2n!/4n!)(4^{2n}/2\sqrt{n})$.

Then two applications of the lemma give:

$$P_{2n} = (n!/2\sqrt{n})e^n n^{-n} A_n / A_{2n} \quad \dots\dots(1)$$

Also $$P_{2n}/P_n = (4^n/\sqrt{2})(2n!/n!)^2(2n!/4n!),$$

and three applications of the lemma reduce this to

$$P_{2n}/P_n = A_n^2/A_{2n}\sqrt{2}. \quad \dots\dots(2)$$

Now *if we assume that* A_n *tends to a definite limit*, as the author does by using a for any A_n, when n is large, (1) gives the required result, independently of (2); for then $A_n/A_{2n} \to 1$, whatever the precise value of $\lim A_n$ may be, while $P_{2n} \to \sqrt{\pi/2}$; and when n is large the close approximation between $n!$ and $\sqrt{2n\pi}\, n^n e^{-n}$ is assured.

On a like assumption, since $P_{2n}/P_n \to 1$, (2) shows that the value of $\lim A_n$ is $\sqrt{2}$. This the original note brings in as the value of a and uses as a step to the final conclusion. If A_n tends to a, a is of course the same thing as $\lim A_n^2/A_{2n}$.

But the important point is that even (2) in itself does not imply or ensure that A_n *has* a definite limit. If A_n were (approximately) $\sqrt{2}\theta^n$, where θ is different from unity, (2) would hold equally well; but in (1) A_n/A_{2n} would then tend to θ^{-n} and the conclusion would not hold. This is an excellent illustration of the traps which beset the unwary!

3. The reasoning will be valid only if we can prove that the product $a_1 a_2 \dots a_n$, in the lemma, tends to a definite limit; and this depends on rather nice considerations.

We have $$\log\left\{\left(1+\frac{1}{n}\right)^{-n}\right\} = -n\left(\frac{1}{n} - \frac{1}{2n^2} + \dots\right) = -1 + \frac{1}{2n} - \dots,$$

$$\therefore \left(1+\frac{1}{n}\right)^{-n} = e^{-1}\left(1+\frac{1}{2n}+\dots\right).$$

The other factor in (b), $(2n+1)/(2n+2)$, is $1 - 1/2n + \dots$.

Thus $a_{n+1} = (1+1/2n\dots)(1-1/2n\dots)$; and it is only because the terms in $1/n$ cancel out and the product differs from unity by terms of the order $1/n^2$ at least that the definite limit, on which the success of the whole depends, is assured. If $a_n \equiv 1 + \alpha_n$, it is not enough that α_n should tend to zero, to secure that $\Pi(a_n)$ should have a finite limit; it is essential that $\Sigma\alpha_n$ should be convergent.

PERCY J. HEAWOOD.

971. [A. 2. b.] *The Solution of a Problem.*

Several different lines have been followed in the solution of the puzzle which was given in the January *Gazette*, p. 20, as Gleaning 718.

If the ages of the three rhinoceri are taken as the variables, it is easily found that these ages r, m, f satisfy the equations:

$$m+rf=r(m+r^2), \quad \ldots\ldots(1)$$

$$137m-126f-18r=84, \quad \ldots\ldots(2)$$

$$11m+2f-13r=275, \quad \ldots\ldots(3)$$

it being assumed that the difference in ages between mother and father is $f-m$, or in other words, that the mother is not older than the father.

One solver eliminates f from the last two equations, and treats the resulting equation

$$830m-837r=17409 \quad \ldots\ldots(4)$$

as indeterminate. The simplest solution, $m=24$, $r=3$, implies $f=25$, and these values evidently satisfy equation (1).

Instead of solving (4) systematically, we need only notice that since m is a multiple of r, r is a factor of 17409, that is, of $3.7.829$, and that since $837r+17409$ is a multiple of 10, the only one of the factors that can lead to a solution of our problem is 3.

Other solvers, ignoring the principle that if a solution has to be in whole numbers, consequent short-circuiting is not merely permissible but right, are more pertinacious, and substitute for m and f from (2) and (3) in (1), obtaining for r the cubic equation

$$1660r^3+91r^2-3607r-34818=0, \quad \ldots\ldots(5)$$

of which $r=3$ is the only positive root, and in fact, as Mr. T. M. A. Cooper points out, the only real root.

Far less laborious arithmetic is involved if the first sentence in the question is recognised to imply that the three ages may be taken as r, nr, r^2+nr-n. With the same assumption that the mother is not the older parent, the second sentence then implies that if q is given by $nr+q=8(r+q)$, that is, by

$$(n-8)r=7q, \quad \ldots\ldots(6)$$

then

$$19nr+12=18(9r+8q+n-r^2), \quad \ldots\ldots(7)$$

and the second paragraph gives the relation

$$13(n-1)r+2(r^2-n)=275. \quad \ldots\ldots(8)$$

From (8), $(n-1)r$ is odd, whence r is odd and n is even. Hence $9r-r^2+n$ is even, whence from (7) n is a multiple of 4. Also r^2-1 is a multiple of 4, and if we write (8) in the form

$$13nr-13(r+1)+2(r^2-1-n)=260, \quad \ldots\ldots(9)$$

we see that $r+1$ is a multiple of 4. Then further, writing $9r-r^2$ as

$$8r+(r+1)-(r^2-1)-2,$$

we see that the expression on the right of (7) is an odd multiple of 4, and therefore that n is a multiple of 8, and finally since 260 is an odd multiple of 4 it follows that $r+1$ also is an odd multiple of 4. Thus we may write

$$r=8s+3, \quad n=8p,$$

where p is to be positive and s is to be positive or zero.

Equation (9) now becomes

$$16s^2+(104p-1)s+37(p-1)=0. \quad \ldots\ldots(10)$$

Obviously if p is greater than 1 and s is positive or zero, the expression on the left of (10) is wholly positive. Hence the one possibility is $p=1$ with the non-negative root $s=0$, giving $r=3$, $n=8$, from which $q=0$; this set of values does satisfy (7), and the existence of one and only one solution is proved.

Prof. Lodge comments: "That Rhino problem is a bit of a fraud. They talk about the time when the mother is 8 times the age of the son and also when the son's age is less by 4 years than 1/7 of the combined ages of the parents. In each case it is the fact *at present*. It makes it tedious to the solver, but fairly easy to the setter provided he can manage the mystification of language needed." Later he writes: "I said 'a bit of a fraud' of course not in a contemptuous sense, but rather of admiration for its ingenuity!"

It is not difficult to examine the assumption as to the relative ages of mother and father. If the difference in ages is interpreted as $m-f$, that is, as $n-r^2$, equations (7), (8) are replaced by

$$19nr+12=18(9r+8q)+10(n-r^2), \quad \ldots\ldots\ldots\ldots\ldots\ldots(7')$$

$$13(n-1)r-6(n-r^2)=275. \quad \ldots\ldots\ldots\ldots\ldots\ldots(8')$$

Arguments precisely parallel to those used previously shew that n and $r+1$ are both multiples of 8, so that we may write

$$r=8s'-1, \quad n=8p',$$

where p' and s' are now both positive. With these substitutions, (8') becomes

$$48s'^2+(104p'-25)s'-(19p'+32)=0,$$

and we may write this equation in the form

$$(48s'^2-25s')+32(p's'-1)+(72s'-19)p'=0,$$

to shew that no solution in positive integers is possible.

The cubic equation for this case is

$$976r^3+91r^2-2951r-19754=0,$$

which has a real root a little greater than 3, but if m and f are calculated from this value of r, it is f which is the greater, in contradiction to the very assumption on which the equation is based.

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784. Some time ago the University of Cambridge was startled by an examiner, who set questions implying that one *magnitude* could be divided by another,—that the Senate unanimously ousted the examiner,—and that the implication was considered to be declared *falsa in philosophiâ et ad minus erronea in fide*. We were, therefore, rather surprised, but much pleased, to see the following question in an examination paper recently given at St. John's College:

"Divide 22,557 days, 20 hours, 20 minutes, 48 seconds, by 57 minutes, 12 seconds." It was also asked to explain the fraction $\frac{3l.\ 18s.\ 8d.}{6l.\ 12s.\ 9d.}$ and to take that fraction of 104 yards 9 inches. . . .

What reparation is to be made to the examiner who was declared a heretic for saying that the thing could be done—which the haberdasher's apprentice does every day of his life? We happen to know that his Cambridge prospects were completely destroyed by the vote of the Senate.—*Athenaeum*, 1858, ii. p. 838.

785. "Not emotion or imagination . . .
not even thatt supersensuous sublimation of thought,
the euristic vision of mathematical trance,
hath any other foundation than the common base of Nature's building."
—Robert Bridges, *The Testament of Beauty*, I. 365-370
[*Per* Miss M. O. Stephens].

786. Buffon, says Mr. D'Israeli, wrote his own panegyric beneath his statue in the library of the *Jardin des Plantes*, and asserted of the great geniuses of modern ages, "there are but five, Bacon, Newton, Leibnitz, Montesquieu and MYSELF."—Quoted in *Athenaeum*, 1854, p. 204.

REVIEWS.

The Earth: Its Origin, History and Physical Constitution. By HAROLD JEFFREYS. Second Edition. Pp. x + 346. £1. 1929. (Cambridge University Press.)

When the first edition of Dr. Jeffreys's *The Earth* appeared in 1924, it was at once apparent that the work would exercise great influence on the future study of the history, interior constitution and large scale surface features of the earth. The task of coordinating knowledge from many widely different fields so as to present an ordered scientific account of the subject was one of no small difficulty, and Dr. Jeffreys, to whom much of the recent progress in the subject was due, performed it in such a manner as to make his book, at once a necessity to the worker in any branch, a most lucid account of an interesting subject for the ordinary scientist, and a model of scientific method.

He has not disappointed us in his second edition. All the changes, both of subject-matter and presentation, are for the better—and they are considerable. The most important change is the place of honour now given to seismology, which enables inferences to be drawn as to the constitution of the interior of the earth, and therefore is discussed before the thermal history. But the general impression left by a comparison of the two editions is that much greater probability and more precision is attained almost everywhere, and that the book most certainly presents the subject as a coherent whole more satisfactorily than the earlier one.

The first four chapters again deal with the origin of members of the solar system, and there is but little change. The Nebular Hypothesis of Laplace is shown to be unsatisfactory in Chap. I, the author's reasons for abandoning the Planetesimal Hypothesis of Chamberlin and Moulton are set out in Appendix A, and a Tidal Theory is given in Chap. II as the only satisfactory alternative. (Jeans's variation of the Tidal Theory, as given in his *Problems of Cosmogony and Steller Dynamics*, receives attention in Appendix B.) Chap. III develops Darwin's Resonance Theory of the origin of the moon. Chap. IV contains an account of the origin, density, motion, evolution and effects of the resisting medium, and from the present eccentricity of the orbit of Mercury finds an estimate of the age of the solar system, of the same order of magnitude as those derived from other sources. The only change of any importance in these chapters is that the primitive sun, instead of being considered very distended, is, in consequence of Eddington's mass luminosity relation and its results, considered to have the same mass, radius and mean density as at present. It is probable that in the next edition Chapter II will be very considerably changed, since the author has now adopted Buffon's hypothesis, of direct collision of the sun and a star (Buffon's 'comet') in place of the tidal theory, mainly because of the greater ease in accounting for the rotations of the sun and planets.* But the main point, that the earth must have passed through a liquid stage, remains unaffected.

Improved determinations of the age of the earth by means of the laws of radioactivity and the uranium : thorium : lead ratios for rocks are given in Chap. V. The improvements arise from better data and an allowance for the thorium, and are mainly due to Holmes. They fix the age between the narrow limits $1{\cdot}3 \times 10^9$ and 3×10^9 years (instead of $1{\cdot}3 \times 10^9$ and 8×10^9 years as before).

Chapters VI and VII are now devoted to seismology, and are almost entirely new. In Chapter VI, after a useful summary of the main necessary results of petrology, the author proceeds to details of a method of discussing near earthquakes, with particular reference to the Jersey and Hereford earthquakes of 1926. It appears that three distinct compressional and three distortional pulses could be recognised. The author's interpretation and reduction of the observations lead him to infer the existence of three layers in the crust, the upper one being about 10 km. and the intermediate one about 20 km. thick. The ratios of the bulk modulus to density for the layers are also given by the seismological evidence, and from these Dr. Jeffreys

*M.N.R.A.S., 89, 636-641 (1929); 89, 731-739 (1929).

infers that the upper layer is probably granite, the intermediate one tachylyte (glassy basalt) and the lowest dunite. Surface waves are considered and shown to give consistent results, and, generally, a good case is made out for the probable truth of the inferences.

The main results of Chapter VII is to establish the existence of a core, of radius about 0·545 times the radius of the earth, with a sharp boundary, and incapable of transmitting distortional waves. Thus the core is either a true liquid or an imperfectly elastic solid absorbing the distortional waves before they penetrate. Which is correct cannot be decided from the seismological evidence, but is decided later, in Chapter XIII, by a consideration of the bodily tide and Love's numbers. No such discussion of the bodily tide was given in the first edition; in fact, according to the preface, it seems to have been deliberately omitted because it was thought that more detailed and deliberate information could be derived from seismology. It is interesting to see how it now provides a most important link in the chain of evidence.

Fortified with this new knowledge of the internal constitution, Dr. Jeffreys gives a new discussion of the thermal history of the earth in Chapter VIII, which includes a new approximate but very general solution of the problem of the cooling of the earth's crust. By assuming that radioactivity in the crust is practically confined to the upper and intermediate layers, and taking the thickness of one of them as known, the other can be found. The result is shown to be in accordance with those derived from the seismological evidence. Also, on the basis of the calculation, a new depth-temperature curve is given, and the cooling at any depth found.

Chapters IX and X, on the equations of motion of an elastic solid with initial stress and the bending of the earth's crust by the weight of mountains, serve mainly as mathematical preliminaries to the theory of isostasy, and are almost unaltered. But the chapter on isostasy is extended, and the results are much more definite, the author being even able to infer theoretically, and fairly convincingly, that large mountain systems occur in regions bounded by faults, the compensation being distributed over the region between the extreme planes of failure. Another result is that the asthenosphere must have a strength of 10^8 dynes/cm.^{2}, a point on which the author becomes more emphatic in Chapter XII, after considering the figure of the moon, and showing that the moon's interior must have a strength comparable with that of the asthenosphere. One factor making for greater definiteness in the theory of isostasy is, of course, that we can now consider the compensation as concentrated at the bases of the granite and intermediate layers.

The main new matter in Chapter XII, on the figures of the earth and the moon, comes from modifying Wiechert's hypothesis, in which the earth is supposed composed of two strata of uniform density, by taking the core as 0·545 times the radius of the earth, as suggested by the seismological evidence. The result is to confirm that the rocky shell is of one material, dunite, from a depth of 30 km. down to the "great discontinuity," and that the central core is of metallic iron.

Chapter XIII contains the discussion of the bodily tide already mentioned, and the original Chapter XV on the variation of latitude, not much altered.* Nor is there much alteration in the chapter on tidal friction. It is found that the persistence of the variation of latitude (free motion) and the absorption of seismic waves together give a measure of the kind of elastic afterworking likely to be found and an upper limit (a low one) to the plasticity, so that the author no longer looks on the persistence of the variation of latitude as a difficulty.

The last chapter, called the Origin of the Earth's Surface Features, is the equivalent of the two chapters on the thermal contraction theory of mountain formation and theories of other surface features in the original edition. The discussion of mountain formation has been largely recast. Oceanic deeps, having received a passing mention in Chapter XII, are not again mentioned, nor are geosynclines. A criticism in the first edition of Osmond Fisher's theory of the origin of the continents is admitted to have been mistaken.

* A new discussion of the forced motion has since been given by L. Rosenhead, M.N.R.A.S. Geophys. Suppl., 2, 140-170 (1929).

Vulcanism and intrusion are dealt with at length, partly perhaps because Holmes has gone to the length of abandoning his and Dr. Jeffreys's theory of the earth's cooling because of the problems raised.

It has been possible to mention only a few of the more interesting points and the book must be read as a whole to watch how the chain of evidence is forged. The great strides made by the science since the first edition are all well set out, and it is to be hoped that as much or even greater progress will be shown in the next five years. There are still many unsolved problems in geophysics.

There is not much one finds fault with in reading the book. The core is suddenly and definitely referred to as the iron core before it is shown or even hinted at that it is iron, and there are a few other anticipations that might prove puzzling to one new to the subject. There are still some references lacking that could be given with advantage, though the defect is marked less than in the first edition. Thus, selecting two cases at random, I may mention that no reference is given to the papers of Vening Meinesz, whose work on the determination of gravity at sea is mentioned on p. 223, and no definite reference to the work on the secular acceleration of the moon in lunar theory is given on p. 255, though we are given a reference for the equations of motion of a rigid body referred to moving axes, which is surely commoner knowledge. I find the chapter on the resisting medium one of the most difficult in the book, and a summary, if possible, would be welcome.

One last point. I remember seeing in Germany a film of the formation of the Dolomites by horizontal compression, folding and denudation. Is it beyond the technique of film makers to film parts of the story of the earth, as Dr. Jeffreys relates it? If not, is there no scientific society which will have it done? The film should be popular enough, even though the sun is the only star in it.

SYDNEY GOLDSTEIN.

Intermediate Dynamics and Properties of Matter. By R. A. HOUSTOUN. Pp. ix+139. 3s. 9d. 1929. (Longmans, Green.)

This book deals in seven chapters (73 pp.) with Elementary Statics and Dynamics; then in four chapters (41 pp.) with Hydrostatics, and ends with a chapter on Properties of Matter (15 pp.) and 4 pp. of examination questions, also Answers.

The book is attractively arranged, and the explanations in the text are clear and interesting, so it is with regret that the reviewer calls attention to what he regards as the serious defect of the book.

There is so much dealt with so briefly that it is doubtful whether a student who had worked conscientiously through the whole book would have a firm grasp of any of it or be able to face even an easy examination on it.

Consider this instance: Uniform Circular Motion, Simple Harmonic Motion and Projectiles are (together) dealt with in 4 pages of text. There are on this four examples for the student to work, two on Circular Motion, two on Simple Harmonic Motion.

As a revision book or one to be supplemented by notes and collections of examples from the teacher, the book might well be very useful.

Examples in Mechanics. By A. ROBSON and C. J. A. TRIMBLE. 4s. 6d. 1930. (Bell and Sons.)

This book, intended for the use of those preparing for examinations of Higher Certificate or Scholarship standard, is by two men highly experienced and successful in this class of teaching, and—especially as a volume of hints for the harder examples is in preparation—should be extremely useful to all engaged in teaching Mechanics to Mathematical Specialists.

The examples range in difficulty from those suitable for commencing specialists who have mastered the course for Additionals up to the harder questions set in Scholarship papers.

The sections on Kinematics (73 ex.), Elementary Dynamics (86 ex.) and Elementary Statics (55 ex.) will be those most useful for first-year specialists, not that all of these examples are by any means easy.

The whole number of examples is 818, of which the last 70 form 10 revision papers.

Among the various sections those on Simple Harmonic Motion (53 ex.), Friction (63 ex.) and Virtual Work (62 ex.) are perhaps especially noteworthy.

As this book was in the press at the same time as Mr. Ramsey's new book on Dynamics, a few of the Dynamics questions, taken from Scholarship papers, occur in both books. There is probably no such collection of examples in Statics available elsewhere.

All those teaching Mechanics to Mathematical Specialists are warmly advised to get this book.

Examples in Applied Mathematics. By R. O. Street. Pp. 160. 4s. 1929. (Methuen & Co.)

This collection of 980 examples is intended primarily for the use of University students and consists mainly of questions which have been set in the examinations of the University of Liverpool. The range is approximately that of a three-year course subsequent to the Intermediate Examination, and the greater part of the usual syllabus for pass and honours degrees in Science and Engineering is covered, the examples being mainly of a difficulty suitable to those reading for an honours degree.

The questions are all set in the form " show that " or " prove that " so that no answers have to be appended to the book.

Sections I, II deal with the motion of a particle in a straight line, and plane. Section IV contains 164 examples on motion of a rigid body in two dimensions, and Section V has 37 miscellaneous problems in three dimensions.

Section VI on Lagrange's Equations and Section VII on motion of a rigid body in three dimensions are more advanced.

With Section XII begins a series of sections dealing with Hydrostatics, Hydrodynamics and Sound, Electricity, Attractions.

This, like the last, should prove to be an extremely useful collection of examples.

The Elements of Mechanics. By W. D. Hills. Pp. viii + 140. 2s. 9d. 1929. (University of London Press.)

This attractive little book gives with as little mathematics as possible a general account of the ideas of motion, force and mass, preparatory to a technical training or to the study preparing for University Matriculation.

It thus covers the easier parts of the elementary course in Mechanics. It also (in a chapter on Material Substance, etc.) gives a short account of the structure of the atom and concludes with a chapter on the measurement of time. The book contains an adequate supply of easy examples.

Throughout the attention of the reader is drawn to real things and he is urged to reflect on familiar examples. The style is lucid, simple and intelligible. The diagrams are a distinct feature both in the text and in the attractive headings to the chapters which are little pictures illustrative of some mechanical principle, not without a touch of humour here and there, as in the picture of the hare racing the tortoise to illustrate velocity and of the recumbent lion labelled " gravity triumphant ".

An excellent book on Mechanics for those beginning the subject early.

C. O. Tuckey.

Eléments de Mathématiques Financières—Opérations à Long Terme. By R. Thiry. Pp. viii + 91. Cm. 22 × 14. Fcs. 16. 1930. (Paris, Vuibert.)

This brochure embodies lectures delivered by the author at the Institute of Higher Commercial Education at Strasbourg, and it may be doubted whether it will have any great interest for English readers. It is intended not for experts, on either the scientific or the financial side, but for business men requiring a general acquaintance with the nature and underlying principles of long-term redeemable loans: the author considers that in these days of international debts and settlements some knowledge of such matters forms, more and more, part of the " *bagage indispensable* " of modern men, whatever their activities. It is also perhaps specially true of such matters that " a little knowledge is a dangerous thing ". The author claims with justice that beyond quite elementary algebra his treatment demands only the (elementary) theory

of fractional and negative exponents and of geometrical progressions, and readers to whom even these things present difficulties will find a short treatment of them in an appendix. Within the limits the author has set himself the work seems to be clearly and agreeably written, from the Continental standpoint. It opens with a brief treatment of compound interest and the valuation of annuities-certain, and proceeds to apply the results to the theory and practice of loans redeemable in various ways. Some short compound interest tables, designed to familiarize the reader with the nature and application of more extensive standard tables, are given at the end of the book.

G. J. L.

Differential Geometry of Three Dimensions. Volume II. By C. E. WEATHERBURN. Pp. xii + 239. 12s. 6d. 1930. (Camb. Univ. Press.)

The second volume of Professor Weatherburn's *Differential Geometry of Three Dimensions* should prove a very useful book to those familiar with the earlier volume. The branches of the subject discussed are not usually included in an honours course in English universities, but students who have studied the subject to degree standard will find this book an excellent introduction to further work. Much of the volume is devoted to subjects to which the author has himself contributed in the last few years, particularly in the theory of families of curves and surfaces, and of small deformations. Other topics are however included, with the result that the two volumes together give an account of most of the principal branches of classical Differential Geometry. An elementary account of Levi-Civita's theory of parallel displacements is also given.

The general method followed is the same as in the earlier volume. Vectors are employed throughout, and the theory of these has been developed to a considerable extent by the author for the purposes of his researches. Two chapters are devoted to these developments and in these the reader will obtain a knowledge which will be sufficient for the requirements of the remainder of the book. The exposition is commendably simple, and once familiar with the methods, the reader will find it easy to follow the author in his application of them to problems in Differential Geometry.

Frequent references are given throughout the book to original papers, chiefly by the author, and a satisfactory index is appended. As in the earlier volume, there is an abundance of examples.

W. V. D. H.

The Size of the Universe. By L. SILBERSTEIN. Pp. viii + 216. 10s. 1930. (Oxford Univ. Press.)

In 1917 Einstein made a slight change in the equations of his general relativity theory, modifying the law of gravitation in empty space from $G_{\mu\nu} = 0$ to $G_{\mu\nu} = \lambda g_{\mu\nu}$, where λ (called the "cosmical constant") is exceedingly minute according to our ordinary scales of measurement. This little term had startling reactions on the immensity of the universe. We who, like Hamlet, had counted ourselves kings of infinite space became "bounded in a nutshell", and astronomers were set speculating. The theme of Dr. Silberstein's book is the nature of this transformation, its meaning, and its consequences for the stellar universe. The story is capably and entertainingly told. There is, of course, no attempt to dispense with mathematics, but it is not of the lengthy laborious kind. The author does good service in showing many of the interesting bypaths of thought which the subject arouses.

If the general fascination of the subject is insufficient to hold our attention, we have the additional stimulus of a conflict. Almost from the beginning there have been two rival "worlds", known as Einstein's and de Sitter's, which have Einstein's revised equation $G_{\mu\nu} = \lambda g_{\mu\nu}$ as common ground. These two worlds exhibit very different phenomena, but the difference between them lies not in their principle of construction but in what must appear to us a rather accidental circumstance, viz. whether Nature was generous or sparing in the amount of matter she has furnished. A large amount of matter will give enough gravitation to enable the world to pull itself together (Einstein); with too little matter the forces of dissolution gain the upper hand (de Sitter). Dr. Silberstein is enthusiastically in favour of de Sitter's world. That is also our

own preference, but we think that some of his arguments against the alternative are invalid. He does not allow for the fact that both Einstein's and de Sitter's worlds are simplified types, which the actual universe might approximate to but obviously does not reach. It has been said that Einstein's world contains matter but no motion, whilst de Sitter's world contains motion but no matter. The actual universe must be to some extent a compromise between them, and either model can be made to appear ridiculous if it is treated as other than a mathematical limit.

The moment chosen for writing this book has proved to be rather unfortunate. The theory had been almost stationary for ten years, but it has suddenly developed a considerable advance mainly owing to the work of Lemaître. It is now unnecessary to confine attention to the two limiting cases, and we can deal quite easily with intermediate forms. Einstein's world turns out to be unstable. It seems probable that the universe started as an Einstein world but could not remain in that form owing to instability, and that it is now on its way towards the de Sitter form which lies ahead as the ultimate limit. This point of view is too recent to appear in the book.

Robbed of our infinitude of space we are anxious to know how much room is left us. It is on this point that Dr. Silberstein is most heterodox, insisting on a much smaller radius than that found by other investigators. We do not learn the worst until we reach his Appendix. In the text he takes leave of us imprisoned for ever in a space of 36 million parsecs radius or a little smaller. He returns in the Appendix to cut down our prison to 2 million parsecs radius. This would be a matter of serious concern to astronomers who have been planting out spiral nebulae somewhat beyond the latter distance. Even if we were convinced of the soundness of his method of determining the constant (which differs from the method generally adopted) we should still object that he has chosen very unsuitable data to exercise it on. His reduced radius of space is based on stellar proper motions; these show many systematic features, arising from the dynamics of the galaxy which they form, and the systematic effects of the cosmical term are likely to be masked. The feature which Dr. Silberstein interprets as indicating a 2 million parsecs radius of space is in fact generally attributed by astronomers to the rotation of the galaxy. A. S. E.

A Course of Analysis. By E. J. Phillips. Pp. viii + 361. 16s. 1930. (Camb. Univ. Press.)

The treatment of inequalities in this book is more adequate than in most English text-books, but the remaining chapters do not come up to the same standard. H. D. U.

CORRESPONDENCE.

Sir,—On receiving the July number of the *Mathematical Gazette*, I turned at once to the correspondence column to see what people had to say on the important issue raised by Mr. Siddons. It is one which primarily concerns schoolmasters, more a matter of Psychology than of Mathematics. Judge then of my surprise when I found that not a single schoolmaster had anything to say about it at all.

Professor Levy points out that most of the modern exponents of rigour were brought up in a less mathematically ascetic school. This does not of course mean that the ways of our benighted forefathers are good enough for our children; but it is a fact to be considered on its merits as showing that lack of rigour in early training does not necessarily prevent the mind from appreciating rigour later on. Setting this alongside what every mathematical master knows, that an overdose of intricate analysis too early does prevent any appreciation either of rigour or of anything else, the case put by Mr. Siddons seems a strong one.

The problem before the schoolmaster is not how near he can attain to complete rigour in his expositions, but how to make the boy *set up for himself* an ideal of rigour. To achieve this is chiefly a question of starting right and of

proceeding gently. If we introduce the idea in such a way that it strikes the boy as an ingenious device invented by schoolmasters for making short, easy proofs long and difficult, then he naturally develops a distaste for it. If on the other hand we begin by showing him Lewis Carroll's "proof" that every triangle is isosceles, he is fascinated. We must follow this up by continually providing pointed examples of "things that don't work" (for instance, a consideration of the expression $\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\ldots+\frac{1}{\sqrt{2n}}$, which increases with n although every term in it decreases, is a useful preliminary to a treatment of $\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{x}{n}\right)^n$. We shall then find that he is not at all "up against" the idea, and will actually begin to insist on rigour of his own accord.

Nevertheless, although rigorous methods may be made not distasteful, I do not believe that the question ought to be allowed to loom too large at school. It is a big step from a discussion of the expression mentioned above to a formal proof of Tannery's Theorem, and it is questionable whether we should attempt to make boys take it. What are the psychological effects of that type of work? Has anyone ever studied the psychology of mathematicians? Is it a fact that they are frequently lacking in self-confidence? Are they nearly always of introvert type? And if so, is it because of their training in analysis?

For my own part I do believe that mathematicians are inclined to be self-critical, and that they are not so successful as other people in setting up their mask to face the world with, but I am certain that in the long run, that is gain rather than loss. A sound psychological development requires an honesty with oneself which not everyone attains, and if we can use mathematics for setting up an ideal of honesty and searching criticism, then we are doing well.

But as I said before, the question must not loom too large at school. We must not lose sight of the subject itself. Professor Neville shows us the right way when he reminds us that the series for the sine is one of the delightful surprises of Mathematics. That is the chief thing about it as far as we schoolmasters are concerned, and if we forget that point of view, our teaching will be dead.

E. H. LOCKWOOD.

Felsted School, 22nd July, 1930.

DEAR SIR,—In response to criticism based on a misunderstanding, I wish to emphasise that the sole object of my article on differential equations which appeared on pp. 99-102 of the *Gazette*, May 1930, was to point out difficulties which attend the use of a general formula given by Forsyth. I should not for a moment advocate this as the practical method [indicated in equation (B) on p. 100]; indeed I said so explicitly in the opening sentence of the last paragraph of my article.—Yours sincerely,

F. UNDERWOOD.

University College, Nottingham,
21st July, 1930.

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787. The student of mathematics who in the course of a single introductory lecture on the calculus completes the differentiation of the function x^n would be a good deal soothed to know that he has covered in an hour a problem which took the generation of Barrow, Newton, and Leibniz about forty years to clear up.—L. Hogben, *The Realist*, Dec. 1929. [Per Prof. E. H. Neville.]

788. Mathematics in its prime, the mathematics of Newton and Lagrange and Laplace, advanced our knowledge like the mental work of a man in his prime: mathematics dealing with imaginary nonentities is like the unintelligible fancies of a dreaming dotard who *has been* learned and profound, but in his old age lets idle imaginations take possession of him.—R. A. Proctor, *Gentleman's Mag.* Jan. 1884 [on Cayley's *Brit. Assoc. Address*, 1883; and v. *Cambridge Review*, Feb. 20, 1884].

THE LIBRARY.

160 CASTLE HILL, READING.

ADDITIONS.

THE Librarian reports gifts as follows:

From Mr. **J. H. Baker:**

C. HUTTON — Course of Mathematics. II (3) - - - - - 1801
Vol. I of this edition would be welcome.

From Sir **A. S. Eddington:**

L. SILBERSTEIN — Size of the Universe - - - - - - - 1930

From Mr. **W. J. Greenstreet*:**

F. KLIEM und G. WOLFF — Archimedes - - - - - - - - - 1927

G. LORIA — Storia delle Matematiche. I - - - - - 1929

J. PLASSMANN — Fixsternbeobachtungen - - - - - - 1927

B. TZSCHIRNER — Wetterkarte und Wettervorhersage - - - - 1927

H. WIELEITNER — Mathematische Quellenbücher. I: Rechnen und Algebra - - - - - - - - - - - 1927

Lesehefte zur Mathematik. I: Aus der griechischen Mathematik (2 vols.) 1926
Chapters from Greek mathematicians, with translation into German by A. Czwalina.

From Mr. **G. J. Lidstone:**

R. THIRY — Eléments de Mathématiques Financières - - 1930

From Sir **Thomas Muir:**

T. MUIR and W. H. METZLER — Theory of Determinants - - - - - - 1930
This treatise is founded by Metzler on Muir's work of 1882, but it is about four times as large as the original.

From Prof. **E. H. Neville:**

L. SILBERSTEIN — Synopsis of Applicable Mathematics, with Tables (1 (1922) rep.) - - - - - - - - - 1923
The first issue bore the title *Bell's Mathematical Tables.*

From Dr. **D. M. Wrinch:**

M. LECAT — Bibliographie de la Relativité - - - - - 1924

H. W. TURNBULL — The Great Mathematicians - - - - - 1929

The following have been bought:

A. DRONKE — Die Kegelschnitte in synthetischer Behandlungsweise - - - - - - - - - - - 1881

W. v. IGNATOWSKY — Vektoranalysis; II (2 i.e. 1 (1910) rep.) - - - 1921
Vol. I was bequeathed in 1928.

E. JÄNICKE und G. SCHURIG — Geschichte des Unterrichts in den mathematischen Lehrfächern in der Volksschule (2) - - - - 1888

M. E. C. JORDAN — Cours d'Analyse. III (2) - - - - - - 1896
Completing the set of this edition.

* Gifts before his death. Particulars of his bequest to the Library will appear as soon as possible.

A. MILINOWSKI Elementar-synthetische Geometrie der gleichseitigen Hyperbel - - - - - - - - - - 1883
Elementar-synthetische Geometrie der Kegelschnitte 1882

THE NORTHUMBRIAN MIRROR:
or, Young Student's Literary & Mathematical Companion;
forming an Introduction to the *Ladies' Diary*, &c.

Mr. Greenstreet's last discovery in a bookseller's catalogue and his last gift to the Library was a set of this rare journal, which is No. 40 of the Minor Serials described by Prof. Archibald in the 200th Number of the *Gazette*. A correction is to be noticed. Vol. 2 of the *Mirror* consisted of Nos. 7-12, Sept. 1838-May 1840, and was complete in 362 pp. including Index. No. 13, Sept. 1840, begins a new volume, of which the title-page is p. 1 of the number and is dated 1840. It is a little odd that our copy and the two which Prof. Archibald was able to examine all have No. 13 bound in with Vol. 2, for although the editor and printer were changed, there seems to have been only the normal interval before the issue of the next part.

It will be noticed that the date on the title-page of a volume of a journal often needs explanation. Sometimes, as in this case, the page is issued with the first number and bears the date of that number. Usually the page bears the date of the last number. But sometimes the page is issued with an index some months after the volume has closed, and it may then bear the date of its own issue, though in that case the period covered by the volume is usually given as well.

THE CATALOGUE.

A *Third List* was distributed with the July number of the *Gazette*, and completes the printed record to the end of 1929. The following corrections should be noted:

For F. CARPENTIER, p. 12, *read* F. LE CARPENTIER, and transfer to p. 34.

N. STRUYCK: *For* Œuvres *read* Œuvres qui se rapportent au Calcul des Chances, à la Statistique et aux Rentes Viagères, tirées des Œuvres Complètes.

The entries under H. MARTIN and C. NEUMANN should be in italics.

On p. 53, under *Abhandlungen* . . . for *Weber* read *Wiener*.

On p. 54, for *Le Congrès de l'Enseignement* . . . *read Le Congrès d'Enseignement* . . .

TWO REQUESTS.

Mr. Greenstreet was a ready lender of books, and no memorandum of his loans has yet been found among his papers. Mrs. Greenstreet will be very grateful if any friends who have books of his will return them to her, at The Woodlands, Burghfield Common.

Among his unbound volumes of periodicals not a few are incomplete, and it is impossible to doubt that this is because the missing numbers were lent. In this case it is the Association that will be the loser unless forgotten loans are brought to mind by Mr. Greenstreet's death, and it is to the Librarian that restoration should be made. The journals that have suffered most are *L'Intermédiaire des Mathématiciens* and *Sphinx-Œdipe*.

PERSONAL NOTE.

On 5th. June the University of Cambridge conferred an honorary degree on Professor EINSTEIN. On the following day Professor Einstein delivered a lecture in University College, Nottingham. The blackboard used by Professor Einstein during the lecture and afterwards signed by him will be varnished and preserved as a memento.

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